

Mathematics G4403. Modern Geometry
Assignment 22

Spring 2012

Due Monday, April 23, 2012

- (1) Let (M, g) be a Riemannian n -manifold with a volume form $\omega_n \in \Omega^n(M)$ compatible with the Riemannian metric g . Let S be an $(n - 1)$ dimensional submanifold of M with a volume form $\omega_{n-1} \in \Omega^{n-1}(S)$ compatible with the Riemannian metric ϕ^*g , where $\phi : S \rightarrow M$ is the inclusion. Then M and S are oriented, and there exists a unique unit normal vector field ν along S such that $\omega_{n-1} = i_\nu\omega_n$. Prove the following statements.
- (a) $d(i_X\omega_n) = (\operatorname{div}X)\omega_n$.
 (b) $\phi^*(i_X\omega_n) = \langle X, \nu \rangle \omega_{n-1}$.
- (2) Let M be an n -dimensional submanifold in an $(n+1)$ -dimensional Riemannian manifold (\bar{M}, \bar{g}) . Let g be the induced Riemannian metric on M . Suppose that $\eta \in C^\infty(M, TM^\perp)$ is a unit normal vector field along M , so that $p := H_\eta$ (defined on page 128 of do Carmo's book) is a symmetric bilinear form on M . Let (x_1, \dots, x_n) be local coordinates on M , and write

$$g = \sum_{i,j} g_{ij} dx_i dx_j, \quad p = \sum_{i,j} p_{ij} dx_i dx_j, \quad \nabla_{\frac{\partial}{\partial x_k}} p = \sum_{i,j,k} p_{ij,k} dx_i dx_j.$$

The norm square $|p|^2$ of p and the trace $\operatorname{tr}p$ of p are smooth functions on M given by

$$|p|^2 = \sum_{i,j} p_{ij} p_{kl} g^{ik} g^{jl}, \quad \operatorname{tr}p = \sum_{i,j} p_{ij} g^{ij}.$$

Let R and \bar{R} denote the scalar curvatures of g and \bar{g} , respectively, and let \bar{Ric} denote the Ricci curvature of \bar{g} (defined as in Chapter 4, Section 4 in do Carmo's book).

(a) Use the Gauss equation to prove that

$$n(n+1)\bar{R} - 2n\bar{Ric}(\eta, \eta) = n(n-1)R + |p|^2 - (\operatorname{tr}p)^2$$

(b) Use the Codazzi's equation to prove that

$$n\bar{Ric}(\eta, \frac{\partial}{\partial x_i}) = - \sum_{j,k} g^{jk} p_{ij,k} + \frac{\partial}{\partial x_i}(\operatorname{tr}p).$$

- (3) Let N be a 4-manifold equipped with a pseudo Riemannian metric $\bar{g}_{\mu\nu}$ of signature $-, +, +, +$ on N . Let $s_{\mu\nu}$ be a symmetric $(0, 2)$ tensor on N such that $\bar{g}_{\mu\nu}(t) := \bar{g}_{\mu\nu} + t s_{\mu\nu}$ is a pseudo Riemannian metric of signature $-, +, +, +$ on N when $t \in (-\epsilon, \epsilon)$, where ϵ is some positive number. Let \bar{R} and $\bar{R}(t)$

denote the scalar curvatures defined by $\bar{g}_{\mu\nu} = \bar{g}_{\mu\nu}(0)$ and $\bar{g}_{\mu\nu}(t)$, respectively; let $\bar{R}_{\mu\nu}, \bar{R}_{\mu\nu}(t)$ denote the Ricci curvatures defined by $\bar{g}_{\mu\nu}$ and $\bar{g}_{\mu\nu}(t)$, respectively. We use Einstein's summation convention: " $\bar{g}^{\mu\nu} s_{\mu\nu}$ " means " $\sum_{\mu,\nu=0}^3 \bar{g}^{\mu\nu} s_{\mu\nu}$ ", etc.

- (a) Prove that $\frac{d}{dt}\Big|_{t=0} \sqrt{-\det(\bar{g}(t))} = \frac{1}{2} \bar{g}^{\mu\nu} s_{\mu\nu} \sqrt{-\det(\bar{g})}$.
 (b) Define a smooth function $\text{Tr}(s)$ on N and a vector field X^μ on N by

$$\text{Tr}(s) := \bar{g}^{\mu\nu} s_{\mu\nu}, \quad X^\mu = (s^{\mu\nu} - \bar{g}^{\mu\nu} \text{Tr}(s))_{,\nu}$$

where $,\nu$ denotes the covariant derivative with respect to $\frac{\partial}{\partial x^\nu}$. Prove that

$$\frac{d}{dt}\Big|_{t=0} \bar{R}(t) = -\bar{R}^{\mu\nu} s_{\mu\nu} + \text{div} X.$$

- (c) Use (a) and (b) to prove that

$$\frac{d}{dt}\Big|_{t=0} \left(\bar{R}(t) \sqrt{-\det(\bar{g}(t))} \right) = \left(-(\bar{R}^{\mu\nu} - \frac{1}{2} \bar{g}^{\mu\nu} \bar{R}) s_{\mu\nu} + \text{div} X \right) \sqrt{-\det(\bar{g})}.$$