Mathematics G4403. Modern Geometry **Spring 2012**

Assignment 20

Due Monday, April 9, 2012

(1) Let G be an *abelian* Lie group, and let $\mathfrak{g}$ be the Lie algebra of G. Let $\pi : P \to M$ be a principal G-bundle together with a connection Γ, and let $\omega \in \Omega^1(P, \mathfrak{g})$ be the connection 1-form of Γ. Let $\Omega(M, x)$ be the space of loops $\gamma : [0,1] \to M$ with $\gamma(0) = \gamma(1) = x$. Given any $u \in \pi^{-1}(x)$, define $\phi_u : \Omega(M, x) \to G$ by $\mathrm{Hol}(\gamma)(u) = u \cdot \phi_u(\gamma)$, where $\mathrm{Hol}(\gamma) : \pi^{-1}(x) \to \pi^{-1}(x)$ is the parallel transport along γ defined by the connection Γ.
Suppose that $\gamma : [0,1] \to M$ is a C^∞ map such that $\gamma(0) = \gamma(1) = x \in M$, and that the image of γ is contained in an open subset U such that $P|_U \to U$ is a trivial principal G-bundle. Then for any $u \in \pi^{-1}(x)$, there exists a cross section $\sigma : U \to P|_U$ such that $\sigma(x) = u$. Show that

$$\phi_u(\gamma) = \exp\big(-\int_0^1 (\sigma \circ \gamma)^* \omega\big).$$

(2) Suppose that a Lie group G acts smoothly on the right on a C^∞ manifold M. For any $g \in G$, define $R_g : M \to M$ by $R_g(p) = p \cdot g$. Define a right G-action on TM by

$$(p, v) \cdot g = (p \cdot g, (dR_g)_p(v)), \quad p \in M, \quad v \in T_pM, \quad g \in G.$$

Define a right G-action on T^*M by

$$(p, \theta) \cdot g = (p \cdot g, \theta \circ (dR_{g^{-1}})_{p\cdot g}), \quad p \in M, \quad \theta \in T_p^*M, \quad g \in G.$$

Verify the following statements.
 (a) If $X \in \mathcal{X}(M) = C^\infty(M, TM)$ then $X \cdot g = (R_g)_* X$ for any $g \in G$.
 (b) If $\alpha \in \Omega^1(M) = C^\infty(M, T^*M)$ then $\alpha \cdot g^{-1} = (R_g)^* \alpha$ for any $g \in G$.

(3) Let $\Gamma = \{H_p \mid p \in S^{2n+1}\}$ be the connection on the principal $U(1)$-bundle $\pi : S^{2n+1} \to \mathbb{P}_n(\mathbb{C})$ defined in Assignment 19 (2). We now specialize to the case $n = 1$. Let $\mathbb{C}$ be oriented by the volume form $dx \wedge dy$, where $z = x + \sqrt{-1}y$ is the complex coordinate on $\mathbb{C}$. We choose an orientation on $\mathbb{P}_1(\mathbb{C})$ such that $f : \mathbb{C} \to \mathbb{P}_1(\mathbb{C})$, $z \mapsto [\frac{1}{\sqrt{1+|z|^2}}, \frac{z}{\sqrt{1+|z|^2}}]$ is orientation preserving. Let $\nu \in \Omega^2(\mathbb{P}_1(\mathbb{C}))$ be the volume form determined by this orientation and the Riemannian metric $\hat{g}$. Let $\Omega = D\omega \in \sqrt{-1}\Omega^2(S^3)$ be the curvature form of ω. Show that $\Omega = \sqrt{-1}c\pi^*\nu$, where ν is the volume form defined by $\hat{g}$, and c is a real constant. Find c.