Mathematics G4403. Modern Geometry **Spring 2012**

Assignment 19

Due Monday, April 2, 2012

(1) Let (M^n, g) be a Riemannian manifold of dimension n. Given any smooth curve $\alpha : (-\epsilon, \epsilon) \to M$, let $p = \alpha(0)$, and let $(e_1, \ldots, e_n)$ be an ordered basis of T_pM. Then $(e_1, \ldots, e_n)$ is a point in $GL(TM)_p$, the fibre of the frame bundle $GL(TM) \to M$. Let $E_i(t)$ be the unique parallel vector field along $\alpha(t)$ with $E_i(0) = e_i$. Let $\tilde{\alpha} : (-\epsilon, \epsilon) \to GL(TM)$ be the smooth curve in $GL(TM)$ defined by $\tilde{\alpha}(t) = (\alpha(t), E_1(t), \ldots, E_n(t))$. We call $\tilde{\alpha}$ the *horizontal lift* of α through $(p, (e_1, \ldots, e_n))$. We define a linear map $L_{(p,(e_1,\ldots,e_n))} : T_pM \to T_{(p,(e_1,\ldots,e_n))}GL(TM)$ as follows. Given $w \in T_pM$, let $\alpha : (-\epsilon, \epsilon) \to M$ be a smooth curve with $\alpha'(0) = w$, and let $\tilde{\alpha} : (-\epsilon, \epsilon) \to TM$ be the horizontal lift of α through $(p, (e_1, \ldots, e_n))$. Define

$$L_{(p,(e_1,\ldots,e_n))}(w) = \tilde{\alpha}'(0).$$

(a) Let $(x_1, \ldots, x_n)$ be local coordinates on an open set $U \subset M$. Given $A = (a_{ij}) \in GL(n, \mathbb{R})$,

$$\Big(\sum_{j=1}^n a_{j1} \frac{\partial}{\partial x_j}\big|_p, \ldots, \sum_{j=1}^n a_{jn} \frac{\partial}{\partial x_j}\big|_p\Big)$$

is a point in $GL(TM)_p$. Then x_i and a_{jk}, where $i, j, k = 1, \ldots, n$, are coordinates on $GL(TU) = GL(TM)|_U$. Define the horizontal $\tilde{X}_i$ of $\frac{\partial}{\partial x_i}$ similar to the definition in (1)(a), so that $\tilde{X}_1, \ldots, \tilde{X}_n$ are vector fields on $GL(TU)$. Express $\tilde{X}_i$ in terms of the following local frame of the rank $(n + n^2)$ vector bundle $T(GL(TM))$:

$$\frac{\partial}{\partial x_j}, \quad \frac{\partial}{\partial a_{kl}}, \quad j, k, l = 1, \ldots, n.$$

(b) Given any $(p, (e_1, \ldots, e_n)) \in GL(TM)$, define the horizontal space $H_{(p,(e_1,\ldots,e_n))}$ to be the image of the linear map

$$L_{(p,(e_1,\ldots,e_n))} : T_pM \to T_{(p,(e_1,\ldots,e_n))}GL(TM).$$

This defines a connection on the principal $GL(n, \mathbb{R})$-bundle $GL(TM) \to M$. Let ω be the connection 1-form, which is a $\mathfrak{gl}(n, \mathbb{R})$-valued 1-form on $GL(TM)$. Write ω as an $n \times n$ matrix $(\omega_{\alpha\beta})$ whose entries $\omega_{\alpha\beta}$ are 1-forms on $GL(TM)$. Express $\omega_{\alpha\beta}$ in terms of the following local frame of $T^*(GL(TM))$:

$$dx_j, \quad da_{kl}, \quad j, k, l = 1, \ldots, n.$$

(2) Let $S^{2n+1} = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \mid |z_0|^2 + \cdots + |z_n|^2 = 1\}$. Write $z_j = x_j + \sqrt{-1}y_j$, where $x_j, y_j \in \mathbb{R}$. Let g be the Riemannian metric

on S^{2n+1} induced from the Euclean metric $g_0 = \sum_{j=0}^{n}(dx_j^2 + dy_j^2)$ on $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$. Let $U(1) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ act on S^{2n+1} on the right by

$$(z_0, \dots, z_n) \cdot \lambda = (z_0\lambda, \dots, z_n\lambda).$$

Then $U(1)$ acts freely, properly, and isometrically on (S^{2n+1}, g). There is a unique Riemannian metric $\hat{g}$ on $\mathbb{P}_n(\mathbb{C}) = S^{2n+1}/U(1)$ such that $\pi : (S^{2n+1}, g) \to (\mathbb{P}_n(\mathbb{C}), \hat{g})$ is a Riemannian submersion. For every $p \in S^{2n+1}$, the horizonal space $H_p \in T_pS^{2n+1}$ is defined to be the orthogonal complement of $T_p(p \cdot U(1))$ in T_pS^{2n+1}, where $p \cdot U(1) = \{p \cdot \lambda \mid \lambda \in U(1)\}$. Then $\Gamma = \{H_p \mid p \in S^{2n+1}\}$ is a connection on the principal $U(1)$-bundle $\pi : S^{2n+1} \to \mathbb{P}_n(\mathbb{C})$.

(a) The connection 1-form ω of Γ is an element in $\Omega^1(S^{2n+1}, \mathfrak{u}(1)) = \sqrt{-1}\Omega^1(S^{2n+1})$. Find ω. [Hint: You may write your answer as $\omega = i^*\omega_0$, where $i : S^{2n+1} \to \mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ is the inclusion, and $\omega_0 \in \sqrt{-1}\Omega^2(\mathbb{R}^{2n+2})$.]

(b) Let $[z_0, \dots, z_n]$ denote $\pi(z_0, \dots, z_n)$. Given any $\phi \in (0, \frac{\pi}{2})$, define $\gamma_\phi : [0, 1] \to \mathbb{P}_n(\mathbb{C})$ by $\gamma_\phi(t) = [\cos\phi, \sin\phi e^{2\pi\sqrt{-1}t}, 0, \dots, 0]$. There exists $a_\phi \in U(1)$ such that $\mathrm{Hol}(\gamma_\phi)(p) = p \cdot a_\phi$ for any $p \in \pi^{-1}(\gamma_\phi(0))$, where the holonomy $\mathrm{Hol}(\gamma_\phi)$ is defined by the connection Γ. Find a_ϕ.