Mathematics G4403. Modern Geometry **Spring 2012**

Assignment 18

Due Monday, March 26, 2012

(1) Let N be an n-dimensional submanifold of $\mathbb{R}^{n+1}$. Suppose that the normal bundle $TN^\perp$ of N in $\mathbb{R}^{n+1}$ is trivial, so that there exists a smooth section η of $TN^\perp$ with $|\eta| = 1$. Then $\phi : N \times \mathbb{R} \to TN^\perp$, $(p,t) \mapsto (p, t\eta(p))$ is a diffeomorphism. Note that $\exp^\perp : TN^\perp \to \mathbb{R}^{n+1}$ is defined on the entire $TN^\perp$ because $\mathbb{R}^{n+1}$ is complete.

Let $H_\eta(x,y) = \langle B(x,y), \eta\rangle$ be the second fundamental form of N in $\mathbb{R}^{n+1}$ with respect to the unit normal vector field η. Then H_η is a symmetric (0,2) tensor on N. Let g be the Riemannian metric on N induced by the Euclean metric g_0 on $\mathbb{R}^{n+1}$. In local coordinates $(u_1, \ldots, u_n)$ on N,

$$g = \sum_{i,j} g_{ij} du_i du_j, \quad H_\eta = \sum_{i,j} h_{ij} du_i du_j.$$

Let $\tilde{g} = (\exp^\perp \circ \phi)^* g_0$, Then $\tilde{g}$ is a symmetric (0,2) tensor on $TN^\perp$.

(a) Show that, in local coordinates $(u_1, \ldots, u_n, t)$ on $TN^\perp$,

$$\begin{aligned} \tilde{g} &= dt^2 + \sum_{i,j} (g_{ij} - 2t h_{ij} + t^2 \sum_k h_{ik} g^{kl} h_{lj}) du_i du_j \\ &= dt^2 + \sum_{i,j} \left(\sum_{k,l} (g_{ki} - t h_{ki}) g^{kl} (g_{lj} - t h_{lj}) \right) du_i du_j \end{aligned}$$

(Hint: see do Carmo page 232–233 Example 4.6.)

(b) Let $k_1, \ldots, k_n$ be the principal curvatures at $p \in N$. Show that

$$\sqrt{\det(\tilde{g})}(p,t) = \sqrt{\det(g)}(p) \cdot \prod_{i=1}^{n} (1 - t k_i).$$

(2) do Carmo page 237–238 Exercies 4.

(3) Let (M^n, g) be a Riemannian manifold of dimension n. Given any smooth curve $\alpha : (-\epsilon, \epsilon) \to M$, let $p = \alpha(0)$, and let $v \in T_pM$ be any tangent vector at p. Let $V(t)$ be the unique parallel vector field along $\alpha(t)$ with $V(0) = v$. Let $\tilde{\alpha} : (-\epsilon, \epsilon) \to TM$ be the smooth curve in TM defined by $\tilde{\alpha}(t) = (\alpha(t), V(t))$. We call $\tilde{\alpha}$ the *horizontal lift* of α through (p, v). We define a linear map $L_{(p,v)} : T_pM \to T_{(p,v)}(TM)$ as follows. Given $w \in T_pM$, let $\alpha : (-\epsilon, \epsilon) \to M$ be a smooth curve with $\alpha'(0) = w$, and

let $\tilde{\alpha} : (-\epsilon, \epsilon) \to TM$ be the horizontal lift of α through (p, v). Define $L_{(p,v)}(w) = \tilde{\alpha}'(0)$.

(a) Let $(x_1, \ldots, x_n)$ be local coordinates on an open set $U \subset M$, and let $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ be coordinates on TU defined as on page 62 of do Carmo. Define vector fields $\tilde{X}_1, \ldots, \tilde{X}_n$ on TU by

$$\tilde{X}_i(p, v) = L_{(p,v)} \left(\frac{\partial}{\partial x_i}\Big|_p \right).$$

Write $\tilde{X}_i$ in terms of the following local frame of the rank $2n$ vector bundle $T(TM)$:

$$\frac{\partial}{\partial x_j}, \quad \frac{\partial}{\partial y_j}, \quad j = 1, \ldots, n.$$

(You answer should involve the Christoffel symbols Γ^l_{jk} of the Riemannian metric g.)

(b) Given any $(p, v) \in TM$, define the horizontal space $H_{(p,v)}$ and the vertical space $V_{(p,v)}$ by

$$H_{(p,v)} = L_{(p,v)}(T_pM), \quad V_{(p,v)} = (di_p)_v(T_v(T_pM)),$$

where $i_p : T_pM \to TM$ is the inclusion. Show that $H_{(p,v)}$ and $V_{(p,v)}$ form rank n subbundles of $T(TM)$, and that

$$T(TM) = H \oplus V, \quad H \cong \pi^*TM, \quad V \cong \pi^*TM,$$

where $\pi : TM \to M$ is the projection.

(c) The projection $T(TM) = H \oplus V \to V$ defines a V-valued 1-form ω on TM. Express ω in terms of the following local frame of the rank $2n^2$ vector bundle $T^*(TM) \otimes V$:

$$\frac{\partial}{\partial y_i} \otimes dx_j, \quad \frac{\partial}{\partial y_i} \otimes dy_j, \quad i, j = 1, \ldots, n.$$