

Assignment 16

Due Monday, February 27, 2012

There are 4 problems in this assignment. Notation:

- [dC] = do Carmo, *Riemannian Geometry*
- [GHL] = Gallot, Hulin, and Lafontaine, *Riemannian Geometry*
- Let Ric be the Ricci curvature of a Riemannian metric g , and let b be a constant. We write $Ric \geq bg$ if $Ric(p)(v, v) \geq bg(p)(v, v)$ for any $p \in M$ and any $v \in T_pM$.

- (1) [dC] page 207–208 Exercise 3.
- (2) [dC] page 208 Exercise 4.
- (3) Let M^n be a Riemannian manifold, and let $p \in M$. Suppose that \exp_p is defined and has no critical points on an open ball $B_r(0) \subset T_pM$ of radius $r > 0$. Fix an orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM . For any unit vector $v \in T_pM$ and any $t \in [0, r)$, let

$$g_{ij}(t, v) = \langle (d \exp_p)_{tv}(e_i), (d \exp_p)_{tv}(e_j) \rangle$$

and let $J(t, v) = \sqrt{\det(g_{ij}(t, v))} > 0$. Note that $J(t, v)$ does not depend on the choice of the orthonormal basis.

Let $\gamma : (-r, r) \rightarrow M$ be a normalized geodesic with $\gamma(0) = p$ and $\gamma'(0) = v \in T_pM$. Given $\rho \in (0, r)$, let $\{E_1, \dots, E_n\}$ be an orthonormal basis of $T_{\gamma(\rho)}M$, where $E_1 = \gamma'(\rho)$. For $i = 2, \dots, n$, let $Y_i(t)$ be the unique Jacobi field along $\gamma(t)$ with $Y_i(0) = 0$ and $Y_i(\rho) = E_i$. Prove the following statements (see [GHL] 3.H.5.)

(a) $J(t, v) = t^{1-n} \frac{|Y_2(t) \wedge \dots \wedge Y_n(t)|}{|w_2 \wedge \dots \wedge w_n|}$, where $w_i = \frac{dY_i}{dt}(0)$.

(b) $\frac{\partial}{\partial t} \log J(t, v) \Big|_{t=\rho} = \sum_{i=2}^n I_\rho(Y_i, Y_i) - \frac{n-1}{\rho}$.

(c) If M has constant sectional curvature $K = b$ then

$$J(t, v) = \left(\frac{s_b(t)}{t}\right)^{n-1}, \quad \frac{\partial}{\partial t} \log J(t, v) = (n-1)\left(\frac{s'_b(t)}{s_b(t)} - \frac{1}{t}\right),$$

where

$$s_b(t) = \begin{cases} \frac{\sin(\sqrt{bt})}{\sqrt{b}}, & b > 0 \\ t, & b = 0 \\ \frac{\sinh(\sqrt{-bt})}{\sqrt{-b}}, & b < 0 \end{cases}$$

- (d) If the Ricci curvature of M satisfies $Ric \geq bg$, where b is a constant, then

$$\sum_{i=2}^n I_\rho(Y_i, Y_i) \leq (n-1) \frac{s'_b(\rho)}{s_b(\rho)}.$$

- (4) Use (3) to prove the following Bishop volume comparison theorem (see [dC] page 220 Remark 2.6 and [GHL] page 169 Theorem 3.101). Let (M, g) be a complete Riemannian manifold with Ricci curvature $Ric \geq bg$, where b is a constant. Let $B_r(p) \subset M$, $B_r(\tilde{p}) \subset \tilde{M}(b)$ be normal balls of radius $r > 0$, where $\tilde{M}(b)$ is the complete, simply connected, Riemannian manifold of constant sectional curvature b , with $\dim \tilde{M}(b) = \dim M$. Then

$$\text{vol} B_r(p) \leq \text{vol} B_r(\tilde{p}).$$