

Assignment 15

Due on Monday, February 20, 2012

- (1) Let g_K be the incomplete Riemannian metric on \mathbb{R}^n defined in Assignment 13 (1):

$$g = \frac{4(dx_1^2 + \cdots + dx_n^2)}{(1 + K \sum_{i=1}^n x_i^2)^2}.$$

- (a) Find $r_K > 0$ such that the exponential map $\exp_0 : B_{r_K}(0) = \{v \in T_0\mathbb{R}^n \mid |v|_g < r_K\} \rightarrow \mathbb{R}^n$ is a diffeomorphism, and describe \exp_0 explicitly.
- (b) Find the pullback Riemannian metric \exp_0^*g on $B_{r_K}(0)$.
- (2) (a) Let (M, g) be a connected complete Riemannian manifold with non-positive sectional curvature. Prove that any homotopy class of paths with fixed ends p and q contains a unique geodesic. [Hint: The Theorem of Hadamard.]
- (b) Find a connected complete Riemannian manifold (M, g) and two distinct points $p, q \in M$ such that there are two different and homotopic geodesics connecting p and q .
- (3) Let (M, g) be a complete Riemannian manifold, and let $N \subset M$ be a closed submanifold of M . Let $p_0 \in M$, $p_0 \notin N$, and let $d(p_0, N)$ be the distance from p_0 to N . Show that there exists a point $q_0 \in N$ such that $d(p_0, q_0) = d(p_0, N)$ and that a minimizing geodesic which joins p_0 to q_0 is orthogonal to N at q_0 .
- (4) do Carmo page 208 Exercise 5.