

Assignment 13

Due on Monday, February 6, 2012

Notation: [dC] = do Carmo, *Riemannian Geometry*

$$SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}.$$

$$SU(1, 1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

$$S^{n-1} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 = 1\}.$$

- (1) Given any positive constant $K > 0$, define a Riemannian metric g_K on \mathbb{R}^n by

$$g = \frac{4(dx_1^2 + \dots + dx_n^2)}{(1 + K \sum_{i=1}^n x_i^2)^2}$$

Prove that:

- (a) (\mathbb{R}^n, g) has constant sectional curvature K .
 - (b) (\mathbb{R}^n, g) is not complete.
- (2) Let (x, y) be coordinates of \mathbb{R}^2 , and let $z = x + iy$.
- (a) Let $H^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ be the upper half plane, equipped with the Riemannian metric

$$g = \frac{dx^2 + dy^2}{y^2} = -\frac{4dzd\bar{z}}{|z - \bar{z}|^2}.$$

Prove that the map $z \mapsto \frac{az + b}{cz + d}$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$, is an isometry of (H^2, g) .

- (b) Let $D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$, equipped with the Riemannian metric

$$h = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2} = \frac{4dzd\bar{z}}{(1 - |z|^2)^2}.$$

Prove that the map $z \mapsto \frac{\alpha z + \beta}{\bar{\beta}z + \bar{\alpha}}$, where $\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(1, 1)$, is an isometry of (D^2, h) .

- (3) [dC] page 181 Exercise 4.
- (4) Prove that a globally Riemannian symmetric space is a complete and homogeneous Riemannian manifold.