Mathematics G4403. Modern Geometry **Spring 2012**

Assignment 12

Due on Monday, January 30, 2012

- In this assignment, all the manifolds are *connected*.
- [dC]= do Carmo, *Riemannian Geometry*.

(1) Let $\mathcal{H}^n = \{(y_1, \ldots, y_n) \in \mathbb{R}^n \mid y_n > 0\}$ be the n-dimensional upper half space. For any $\alpha \in \mathbb{R}$, define a Riemannian metric g_α on $\mathcal{H}^n$ by

$$g_\alpha = y_n^\alpha (dy_1^2 + \cdots + dy_n^2).$$

By [dC] Chapter 8 Section 3, $(\mathcal{H}^n, g_{-2})$ is a complete Riemannian manifold. Prove that the Riemannian manifold $(\mathcal{H}^n, g_\alpha)$ is complete if and only if $\alpha = -2$. (Hint: consider the geodesic $\gamma(t)$ with $\gamma(0) = (0, \ldots, 0, 1)$ and $\gamma'(0) = \frac{\partial}{\partial y_n}$.)

(2) Let M be a Riemannian manifold with non-positive sectional curvature. Prove that, for any $p \in M$, the conjugate locus $C(p)$ is empty. (Hint: See [dC] page 119-120, Exercise 3.)

(3) A Riemannian manifold M is said to be *homogeneous* if given $p, q \in M$ there exists an isometry of M which takes p into q. Prove that any homogeneous Riemannian manifold is complete.

(4) Show that the point $p = (0, 0, 0)$ of the paraboloid

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2\}$$

is a pole of S and, nevertheless, the sectional curvature of S is positive.