Mathematics G4403. Modern Geometry **Spring 2010**

Assignment 23

Due Monday, April 26, 2010

- [GHL] = S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian Geometry*.
- [PT] = T. Parker and C.H. Taubes, "On Witten's Proof of the Positive Energy Theorem," Comm. Math. Phys. **84**, 223-238.
- For Problem (2) and (3), we use the notation in [PT] Section 2, Section 3. (In [PT], ∇ is the Levi-Civta connection on the spacetime N^4, and $\bar{\nabla}$ is the Levi-Civita connection on the spacelike hypersurface M^3.)
- Greek letters $\alpha, \beta, \ldots \in \{0,1,2,3\}$. English letters $i, j, k, \cdots \in \{1,2,3\}$.

(1) Let (M, g) be a Riemannian n-manifold with a volume form $\omega_n \in \Omega^n(M)$ compactible with the Riemannian metric g. Let S be an $(n-1)$ dimensional submanifold of M with a volume form $\omega_{n-1} \in \Omega^{n-1}(S)$ compatible with the Riemannian metric $\phi^* g$, where $\phi : S \to M$ is the inclusion. Then M and S are oriented, and there exists a unique unit normal vector field ν along S such that $\omega_{n-1} = i_\nu \omega_n$. Prove the following statements.

(a) $d(i_X \omega_n) = (\mathrm{div} X)\omega_n$.

(b) $\phi^*(i_X \omega_n) = \langle X, \nu\rangle \omega_{n-1}$.

(Hint: see [GHL] Section 4.A.2.)

(2) Let $\cdot$ denote the Clifford multiplication, so that

$$e^\alpha \cdot e^\beta \cdot = -e^\beta \cdot e^\alpha \cdot \quad \text{if } \alpha \neq \beta, \text{ and } -e_0 \cdot e_0 \cdot = e_i \cdot e_i \cdot = -Id_S,$$

where $Id_S : S \to S$ is the identity map.

(a) For a fixed point $p \in M$, find the eigenvalues (with multiplicities) linear map $T_{00} + T_{0i} e^0 \cdot e^i \cdot : S \to S$.

(b) Prove that

$$-\frac{1}{8} R_{\alpha\beta ij} e^i \cdot e^j \cdot e^\alpha \cdot e^\beta \cdot = \frac{1}{4}(R + 2R_{00} + 2R_{0j} e^0 \cdot e^j \cdot)$$

Hint: it might be useful to consider the following cases, where i, j, k are distinct: R_{ijik}, R_{ijki}, etc.; R_{ijij}, R_{ijji}; R_{0ijk}, R_{i0jk}; R_{0iji}, etc.

(3) Let $\{e_1, e_2, e_3\}$ be a local orthonormal frame of TM, and let $\{e^1, e^2, e^3\}$ be local orthonormal coframe of T^*M. For any $\phi \in \Gamma(S)$, we have

$$\nabla_i \phi = \bar{\nabla}_i \phi - \frac{1}{2} h_{ij} e^0 \cdot e^j \cdot \phi, \qquad (1)$$

where $\nabla_i := \nabla_{e_i}$ and $\bar{\nabla}_i := \bar{\nabla}_{e_i}$.

(a) Let $\mathcal{D} = e^i \cdot \nabla_i$ be the hypersurface Dirac operator. Use (1) to prove that

$$\mathcal{D}\phi = e^i \cdot \bar{\nabla}_i \phi - \frac{1}{2}(\mathrm{tr} h) e^0 \cdot \phi.$$

(b) Given $\phi, \psi \in \Gamma(S)$, define a complexified vector field X on M by $\alpha(X) = \langle \phi, \alpha \cdot \psi\rangle$ for any $\alpha \in \Omega^1(M)$. Prove that

$$\langle \phi, \mathcal{D}\psi\rangle - \langle \mathcal{D}\phi, \psi\rangle = \mathrm{div} X.$$

Hint: the following facts might be useful:

(i) $\bar{\nabla}$ is compatible with $\langle \ , \ \rangle$: $e_i\langle \phi_1, \phi_2\rangle = \langle \bar{\nabla}_i \phi_1, \phi_2\rangle + \langle \phi_1, \bar{\nabla}_i \phi_2\rangle$.

(ii) $e_0 \cdot$ is Hermitian and $e_1 \cdot$, $e_2 \cdot$, $e_3 \cdot$ are skew-Hermitian with respect to $\langle \ , \ \rangle$.