Mathematics G4403. Modern Geometry **Spring 2010**

Assignment 22

Due Monday, April 19, 2010

(1) Let E be a C^∞ complex vector bundle over a C^∞ manifold M, and let ∇ be a connection on E. The induced connection on $\mathrm{End}(E) = E^* \otimes E$ satisfies

$$\nabla_X(\phi(s)) = (\nabla_X \phi)(s) + \phi(\nabla_X s)$$

for all $\phi \in C^\infty(M, \mathrm{End}(E))$, $X \in \mathcal{X}(M)$, $s \in C^\infty(M, E)$. The exterior covariant derivative $d_\nabla : \Omega^r(M, \mathrm{End}(E)) \to \Omega^{r+1}(M, \mathrm{End}(E))$ satisfies

$$d_\nabla(\alpha \otimes \phi) = d\alpha \otimes \phi + (-1)^r \alpha \wedge \nabla\phi$$

for all $\alpha \in \Omega^k(M)$, $\phi \in C^\infty(M, \mathrm{End}(E))$. The trace map $\mathrm{End}(E) \to \underline{\mathbb{C}}$, where $\underline{\mathbb{C}} = M \times \mathbb{C}$ is the product (trivial) complex line bundle over M, induces $\mathrm{tr} : \Omega^k(M, \mathrm{End}(E)) \to \Omega^{k+1}(M, \underline{\mathbb{C}})$. Prove that if $\phi \in \Omega^k(M, \mathrm{End}(E))$ then

$$d(\mathrm{tr}\phi) = \mathrm{tr}(d_\nabla \phi).$$

(2) Let (E, h) be a hermitian vector bundle over a C^∞ manifold M, and let ∇ be a unitary connection. Let $F \in \Omega^2(M, \mathrm{End}(E))$ be the curvature 2-form of ∇.

(a) Use (1) and the Bianchi identity to show that

$$\mathrm{ch}_k(E, \nabla) := \mathrm{tr}\Big(\big(\frac{\sqrt{-1}}{2\pi} F\big)^k\Big)$$

is a closed $2k$ form for any positive integer $k \leq \dim M/2$.

(b) Let $r = \mathrm{rank}_{\mathbb{C}} E$, and let $P_k : \mathfrak{gl}(r, \mathbb{C}) \to \mathbb{C}$ be any any invariant polynomial, homogeneous of degree k. Use (a) to prove that

$$P_k(E, \nabla) := P_k\big(\frac{\sqrt{-1}}{2\pi} F\big)$$

is a closed $2k$ form for any positive integer $k \leq \dim M/2$.

(3) Let (E, h) be a hermitian vector bundle over a C^∞ manifold M, and let ∇ be a unitary connection. We also use ∇ to denote the induced connection on the dual vector bundle E^*, and on $\Lambda^k E$. The determinant line bundle of E is defined to be $\det E := \Lambda^r E$, where $r = \mathrm{rank}_{\mathbb{C}} E$. Prove the following identities of Chern forms.

(a) $c_k(E^*, \nabla) = (-1)^k c_k(E, \nabla)$.

(b) $c_1(E, \nabla) = c_1(\det E, \nabla)$.

(4) Let M be an n-dimensional submanifold in an $(n+1)$-dimensional Riemannian mannifold $(\bar{M}, \bar{g})$. Let g be the induced Riemannian metric on M. Suppose that $\eta \in C^\infty(M, TM^\perp)$ is a unit normal vector field along M, so that $p := H_\eta$ (defined on page 128 of do

Carmo's book) is a symmetric bilinear from on M. Let $(x_1, \ldots, x_n)$ be local coodinates on M, and write

$$g = \sum_{i,j} g_{ij} dx_i dx_j, \quad p = \sum_{i,j} p_{ij} dx_i dx_j,$$
$$\nabla_{\frac{\partial}{\partial x_k}} p = \sum_{i,j,k} p_{ij,k} dx_i dx_j.$$

The norm square $|p|^2$ of p and the trace $\mathrm{tr}p$ of p are smooth functions on M given by

$$|p|^2 = \sum_{i,j} p_{ij} p_{kl} g^{ik} g^{jl}, \quad \mathrm{tr}p = \sum_{i,j} p_{ij} g^{ij}.$$

Let R and $\bar{R}$ denote the scalar curvatures of g and $\bar{g}$, respectively, and let $\overline{Ric}$ denote the Ricci curvature of $\bar{g}$ (defined as in Chapter 4, Section 4 in do Carmo's book).

(a) Use the Gauss equation to prove that

$$n(n+1)\bar{R} - 2n\overline{Ric}(\eta, \eta) = n(n-1)R + |p|^2 - (\mathrm{tr}p)^2$$

(b) Use the Codazzi's equation to prove that

$$n\overline{Ric}(\eta, \frac{\partial}{\partial x_i}) = -\sum_{j,k} g^{jk} p_{ij,k} + \frac{\partial}{\partial x_i} \mathrm{tr}p.$$