Mathematics G4403. Modern Geometry **Spring 2010**

Assignment 21

Due Monday, April 12, 2010

(1) Let $S^{2n+1} = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \mid |z_0|^2 + \cdots + |z_n|^2 = 1\}$. Write $z_j = x_j + \sqrt{-1}y_j$, where $x_j, y_j \in \mathbb{R}$. Let g be the Riemannian metric on S^{2n+1} induced from the Euclean metric $g_0 = \sum_{j=0}^{n} dx_j^2 + dy_j^2$ on $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$. Let $U(1) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ act on S^{2n+1} on the right by

$$(z_0, \ldots, z_n) \cdot \lambda = (z_0\lambda, \ldots, z_n\lambda).$$

Then $U(1)$ acts freely, properly, and isometrically on (S^{2n+1}, g). There is a unique Riemannian metric $\hat{g}$ on $\mathbb{P}_n(\mathbb{C}) = S^{2n+1}/U(1)$ such that $\pi : (S^{2n+1}, g) \to (\mathbb{P}_n(\mathbb{C}), \hat{g})$ is a Riemannian submersion. For every $p \in S^{2n+1}$, the horizonal space $H_p \in T_pS^{2n+1}$ is defined to be the orthogonal complement of $T_p(p \cdot U(1))$ in T_pS^{2n+1}, where $p \cdot U(1) = \{p \cdot \lambda \mid \lambda \in U(1)\}$. Then $\Gamma = \{H_p \mid p \in S^{2n+1}\}$ is a connection on the principal $U(1)$-bundle $\pi : S^{2n+1} \to \mathbb{P}_n(\mathbb{C})$.

(a) The connection 1-form ω of Γ is an element in $\Omega^1(S^{2n+1}, \mathfrak{u}(1)) = \sqrt{-1}\Omega^1(S^{2n+1})$. Find ω. [Hint: You may write your answer as $\omega = i^*\omega_0$, where $i : S^{2n+1} \to \mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ is the inclusion, and $\omega_0 \in \sqrt{-1}\Omega^2(\mathbb{R}^{2n+2})$.]

(b) Let $\mathbb{C}$ be oriented by the volume form $dx \wedge dy$, where $z = x + \sqrt{-1}y$ is the complex coordinate on $\mathbb{C}$. We choose an orientation on $\mathbb{P}_1(\mathbb{C})$ such that $f : \mathbb{C} \to \mathbb{P}_1(\mathbb{C})$, $z \mapsto [\frac{1}{\sqrt{1+|z|^2}}, \frac{z}{\sqrt{1+|z|^2}}]$ is orientation preserving. Let $\nu \in \Omega^2(\mathbb{P}_1(\mathbb{C}))$ be the volume form determined by this orientation and the Riemannian metric $\hat{g}$. Let $\Omega = D\omega \in \sqrt{-1}\Omega^2(S^3)$ be the curvature form of ω. Show that $\Omega = \sqrt{-1}c\pi^*\nu$, where ν is the volume form defined by $\hat{g}$, and c is a real constant. Find c.

(c) Let $[z_0, \ldots, z_n]$ denote $\pi(z_0, \ldots, z_n)$. Given any $\phi \in (0, \frac{\pi}{2})$, define $\gamma_\phi : [0, 1] \to \mathbb{P}_n(\mathbb{C})$ by $\gamma_\phi(t) = [\cos\phi, \sin\phi e^{2\pi\sqrt{-1}t}, 0, \ldots, 0]$. There exists $a_\phi \in U(1)$ such that $\mathrm{Hol}(\gamma_\phi)(p) = p \cdot a_\phi$ for any $p \in \pi^{-1}(\gamma_\phi(0))$, where the holonomy $\mathrm{Hol}(\gamma_\phi)$ is defined by the connection Γ. Find a_ϕ.

(2) Suppose that a Lie group G acts smoothly on the right on a C^∞ manfold M. For any $g \in G$, define $R_g : M \to M$ by $R_g(p) = p \cdot g$. Define a right G-action on TM by

$$(p, v) \cdot g = (p \cdot g, (dR_g)_p(v)), \quad p \in M, \quad v \in T_pM, \quad g \in G.$$

Define a right G-action on T^*M by

$$(p, \theta) \cdot g = (p \cdot g, \theta \circ (dR_{g^{-1}})_{p \cdot g}), \quad p \in M, \quad \theta \in T_p^*M, \quad g \in G.$$

Verify the following statements.

(a) If $X \in \mathcal{X}(M) = C^\infty(M, TM)$ then $X \cdot g = (R_g)_*X$ for any $g \in G$.

(b) If $\alpha \in \Omega^1(M) = C^\infty(M, T^*M)$ then $\alpha \cdot g^{-1} = (R_g)^*\alpha$ for any $g \in G$.