Mathematics G4403. Modern Geometry **Spring 2010**

Assignment 20

Due Monday, March 29, 2010

(1) Let (M^n, g) be a Riemannian manifold of dimension n. Given any smooth curve $\alpha : (-\epsilon, \epsilon) \to M$, let $p = \alpha(0)$, and let $v \in T_pM$ be any tangent vector at p. Let $V(t)$ be the unique parallel vector field along $\alpha(t)$ with $V(0) = v$. Let $\tilde{\alpha} : (-\epsilon, \epsilon) \to TM$ be the smooth curve in TM defined by $\tilde{\alpha}(t) = (\alpha(t), V(t))$. We call $\tilde{\alpha}$ the *horizontal lift* of α through (p, v). We define a linear map $L_{(p,v)} : T_pM \to T_{(p,v)}(TM)$ as follows. Given $w \in T_pM$, let $\alpha : (-\epsilon, \epsilon) \to M$ be a smooth curve with $\alpha'(0) = w$, and let $\tilde{\alpha} : (-\epsilon, \epsilon) \to TM$ be the horizontal lift of α through (p, v). Define $L_{(p,v)}(w) = \tilde{\alpha}'(0)$.

(a) Let $(x_1, \ldots, x_n)$ be local coordinates on an open set $U \subset M$, and let $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ be coordinates on TU defined as on page 62 of do Carmo. Define vector fields $\tilde{X}_1, \ldots, \tilde{X}_n$ on TU by

$$\tilde{X}_i(p, v) = L_{(p,v)}\left(\frac{\partial}{\partial x_i}\Big|_p\right).$$

Write $\tilde{X}_i$ in terms of the following local frame of the rank $2n$ vector bundle $T(TM)$:

$$\frac{\partial}{\partial x_j}, \quad \frac{\partial}{\partial y_j}, \quad j = 1, \ldots, n.$$

(You answer should involve the Christoffel symbols Γ^l_{jk} of the Riemannian metric g.)

(b) Given any $(p, v) \in TM$, define the horizontal space $H_{(p,v)}$ and the vertical space $V_{(p,v)}$ by

$$H_{(p,v)} = L_{(p,v)}(T_pM), \quad V_{(p,v)} = (di_p)_v(T_v(T_pM)),$$

where $i_p : T_pM \to TM$ is the inclusion. Show that $H_{(p,v)}$ and $V_{(p,v)}$ form rank n subbundles of $T(TM)$, and that

$$T(TM) = H \oplus V, \quad H \cong \pi^*TM, \quad V \cong \pi^*TM,$$

where $\pi : TM \to M$ is the projection.

(c) The projection $T(TM) = H \oplus V \to V$ defines a V-valued 1-form ω on TM. Express ω in terms of the following local frame of the rank $2n^2$ vector bundle $T^*(TM) \otimes V$:

$$\frac{\partial}{\partial y_i} \otimes dx_j, \quad \frac{\partial}{\partial y_i} \otimes dy_j, \quad i, j = 1, \ldots, n.$$

(2) Let (M^n, g) be a Riemannian manifold of dimension n. Given any smooth curve $\alpha : (-\epsilon, \epsilon) \to M$, let $p = \alpha(0)$, and let $(e_1, \dots, e_n)$ be an ordered basis of T_pM. Then $(e_1, \dots, e_n)$ is a point in $GL(TM)_p$, the fibre of the frame bundle $GL(TM) \to M$. Let $E_i(t)$ be the unique parallel vector field along $\alpha(t)$ with $E_i(0) = e_i$. Let $\tilde{\alpha} : (-\epsilon, \epsilon) \to GL(TM)$ be the smooth curve in $GL(TM)$ defined by $\tilde{\alpha}(t) = (\alpha(t), E_1(t), \dots, E_n(t))$. We call $\tilde{\alpha}$ the *horizontal lift* of α through $(p, (e_1, \dots, e_n))$. We define a linear map $L_{(p,(e_1,\dots,e_n))} : T_pM \to T_{(p,(e_1,\dots,e_n))}GL(TM)$ as follows. Given $w \in T_pM$, let $\alpha : (-\epsilon, \epsilon) \to M$ be a smooth curve with $\alpha'(0) = w$, and let $\tilde{\alpha} : (-\epsilon, \epsilon) \to TM$ be the horizontal lift of α through $(p, (e_1, \dots, e_n))$. Define

$$L_{(p,(e_1,\dots,e_n))}(w) = \tilde{\alpha}'(0).$$

(a) Let $(x_1, \dots, x_n)$ be local coordinates on an open set $U \subset M$. Given $A = (a_{ij}) \in GL(n, \mathbb{R})$,

$$\Big(\sum_{j=1}^{n} a_{j1} \frac{\partial}{\partial x_j}\Big|_p, \dots, \sum_{j=1}^{n} a_{jn} \frac{\partial}{\partial x_j}\Big|_p\Big)$$

is a point in $GL(TM)_p$. Then x_i and a_{jk}, where $i, j, k = 1, \dots, n$, are coordinates on $GL(TU) = GL(TM)|_U$. Define the horizontal $\tilde{X}_i$ of $\frac{\partial}{\partial x_i}$ similar to the definition in (1)(a), so that $\tilde{X}_1, \dots, \tilde{X}_n$ are vector fields on $GL(TU)$. Express $\tilde{X}_i$ in terms of the following local frame of the rank $(n + n^2)$ vector bundle $T(GL(TM))$:

$$\frac{\partial}{\partial x_j}, \quad \frac{\partial}{\partial a_{kl}}, \quad j, k, l = 1, \dots, n.$$

(b) Given any $(p, (e_1, \dots, e_n)) \in GL(TM)$, define the horizontal space $H_{(p,(e_1,\dots,e_n))}$ to be the image of the linear map

$$L_{(p,(e_1,\dots,e_n))} : T_pM \to T_{(p,(e_1,\dots,e_n))}GL(TM).$$

This defines a connection on the principal $GL(n, \mathbb{R})$-bundle $GL(TM) \to M$. Let ω be the connection 1-form, which is a $\mathfrak{gl}(n, \mathbb{R})$-valued 1-form on $GL(TM)$. Write ω as an $n \times n$ matrix $(\omega_{\alpha\beta})$ whose entries $\omega_{\alpha\beta}$ are 1-forms on $GL(TM)$. Express $\omega_{\alpha\beta}$ in terms of the following local frame of $T^*(GL(TM))$:

$$dx_j, \quad da_{kl}, \quad j, k, l = 1, \dots, n.$$