Mathematics G4403. Modern Geometry **Spring 2010**

Assignment 19

You do not have to submit solutions to this assignment.

(1) Let N be an n-dimensional submanifold of $\mathbb{R}^{n+1}$. Suppose that the normal bundle $TN^\perp$ of N in $\mathbb{R}^{n+1}$ is trivial, so that there exists a smooth section η of $TN^\perp$ with $|\eta| = 1$. Then $\phi : N \times \mathbb{R} \to TN^\perp$, $(p,t) \mapsto (p, t\eta(p))$ is a diffeomorphism. Note that $\exp^\perp : TN^\perp \to \mathbb{R}^{n+1}$ is defined on the entire $TN^\perp$ because $\mathbb{R}^{n+1}$ is complete.

Let $H_\eta(x,y) = \langle B(x,y), \eta \rangle$ be the second fundamental form of N in $\mathbb{R}^{n+1}$ with respect to the unit normal vector field η. Then H_η is a symmetric (0,2) tensor on N. Let g be the Riemannian metric on N induced by the Euclidean metric g_0 on $\mathbb{R}^{n+1}$. In local coordinates $(u_1, \ldots, u_n)$ on N,

$$g = \sum_{i,j} g_{ij} du_i du_j, \quad H_\eta = \sum_{i,j} h_{ij} du_i du_j.$$

Let $\tilde{g} = (\exp^\perp \circ \phi)^* g_0$, Then $\tilde{g}$ is a symmetric (0,2) tensor on $TN^\perp$.

(a) Show that, in local coordinates $(u_1, \ldots, u_n, t)$ on $TN^\perp$,

$$\begin{aligned} \tilde{g} &= dt^2 + \sum_{i,j} (g_{ij} - 2t h_{ij} + t^2 \sum_k h_{ik} g^{kl} h_{lj}) du_i du_j \\ &= dt^2 + \sum_{i,j} \left(\sum_{k,l} (g_{ki} - t h_{ki}) g^{kl} (g_{lj} - t h_{lj}) \right) du_i du_j \end{aligned}$$

(Hint: see do Carmo page 232–233 Example 4.6.)

(b) Let $k_1, \ldots, k_n$ be the principal curvatures at $p \in N$. Show that

$$\sqrt{\det(\tilde{g})}(p,t) = \sqrt{\det(g)}(p) \cdot \prod_{i=1}^n (1 - t k_i).$$

(2) do Carmo page 237–238 Exercies 4.