Mathematics G4403. Modern Geometry **Spring 2010**

Assignment 18

Due Monday, March 8, 2010

There are 4 problems in this assignment. Notation:

- [dC] = do Carmo, *Riemannian Geometry*
- [GHL] = Gallot, Hulin, and Lafontaine, *Riemannian Geometry*
- Let Ric be the Ricci curvature of a Riemannian metric g, and let b be a constant. We write $Ric \geq bg$ if $Ric(p)(v,v) \geq bg(p)(v,v)$ for any $p \in M$ and any $v \in T_pM$.

(1) [dC] page 120–121 Exercise 4. (This is used in the proof on page 223.)

(2) [dC] page 237 Exercise 3.

(3) Let M^n be a Riemannian manifold, and let $p \in M$. Suppose that $\exp_p$ is defined and has no critical points on an open ball $B_r(0) \subset T_pM$ of radius $r > 0$. Fix an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM. For any unit vector $v \in T_pM$ and any $t \in [0, r)$, let

$$g_{ij}(t,v) = \langle (d\exp_p)_{tv}(e_i), (d\exp_p)_{tv}(e_j)\rangle$$

and let $J(t,v) = \sqrt{\det(g_{ij}(t,v))} > 0$. Note that $J(t,v)$ does not depend on the choice of the orthonormal basis.

Let $\gamma : (-r, r) \to M$ be a normalized geodesic with $\gamma(0) = p$ and $\gamma'(0) = v \in T_pM$. Given $\rho \in (0, r)$, let $\{E_1, \ldots, E_n\}$ be an orthonormal basis of $T_{\gamma(\rho)}M$, where $E_1 = \gamma'(\rho)$. For $i = 2, \ldots, n$, let $Y_i(t)$ be the unique Jacobi field along $\gamma(t)$ with $Y_i(0) = 0$ and $Y_i(\rho) = E_i$. Prove the following statements (see [GHL] 3.H.5.)

(a) $J(t,v) = t^{1-n}\dfrac{|Y_2(t) \wedge \ldots \wedge Y_n(t)|}{|w_2 \wedge \cdots \wedge w_n|}$, where $w_i = \frac{DY_i}{dt}(0)$.

(b) $\dfrac{\partial}{\partial t}\log J(t,v)\Big|_{t=\rho} = \displaystyle\sum_{i=2}^{n} I_\rho(Y_i, Y_i) - \frac{n-1}{\rho}$.

(c) If M has constant sectional curvature $K = b$ then

$$J(t,u) = \Big(\frac{s_b(t)}{t}\Big)^{n-1}, \quad \frac{\partial}{\partial t}\log J(t,v) = (n-1)\Big(\frac{s_b'(t)}{s_b(t)} - \frac{1}{t}\Big),$$

where

$$s_b(t) = \begin{cases} \sin(\sqrt{b}t), & b > 0 \\ t, & b = 0 \\ \sinh(\sqrt{-b}t), & b < 0 \end{cases}.$$

(d) If the Ricci curvature of M satisfies $Ric \geq bg$, where b is a constant, then

$$\sum_{i=2}^{n} I_\rho(Y_i, Y_i) \leq (n-1)\frac{s_b'(\rho)}{s_b(\rho)}.$$

(4) Use (3) to prove the following Bishop volume comparison theorem (see [dC] page 220 Remark 2.6 and [GHL] page 169 Theorem 3.101). Let (M, g) be a complete Riemannian manifold with Ricci curvature $Ric \geq bg$, where b is a constant. Let $B_r(p) \subset M$, $B_r(\tilde{p}) \subset \tilde{M}(b)$ be normal balls of radius $r > 0$, where $\tilde{M}(b)$ is the complete, simply connected, Riemannian manifold of constant sectional curvature b, with $\dim \tilde{M}(b) = \dim M$. Then

$$\mathrm{vol} B_r(p) \leq \mathrm{vol} B_r(\tilde{p}).$$