Mathematics G4403. Modern Geometry **Spring 2010**

Assignment 14

Due on Monday, February 8, 2010

Notation: [dC] = do Carmo, *Riemannian Geometry*

$$\begin{aligned} SL(2,\mathbb{R}) &= \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{R},\ ad-bc=1\}. \\ SU(1,1) &= \{\begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} \mid \alpha,\beta \in \mathbb{C},\ |\alpha|^2-|\beta|^2=1\}. \\ S^{n-1} &= \{(x_1,\dots,x_n)\in\mathbb{R}^n \mid x_1^2+\cdots+x_n^2=1\}. \end{aligned}$$

(1) Given any positive constant $K>0$, define a Riemannian metric g_K on $\mathbb{R}^n$ by
$$g = \frac{4(dx_1^2+\cdots+dx_n^2)}{(1+K\sum_{i=1}^n x_i^2)^2}$$
Prove that:
  (a) $(\mathbb{R}^n, g)$ has constant sectional curvature K.
  (b) $(\mathbb{R}^n, g)$ is not complete.

(2) Let (x,y) be coorindates of $\mathbb{R}^2$, and let $z = x+iy$.
  (a) Let $H^2 = \{(x,y)\in\mathbb{R}^2 \mid y>0\}$ be the upper half plane, equipped with the Riemannian metric
  $$g = \frac{dx^2+dy^2}{y^2} = -\frac{4dzd\bar z}{|z-\bar z|^2}.$$
  Prove that the map $z\mapsto \dfrac{az+b}{cz+d}$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2,\mathbb{R})$, is an isometry of (H^2,g).
  (b) Let $D^2=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2<1\}$, equipped with the Riemannian metric
  $$h = \frac{4(dx^2+dy^2)}{(1-x^2-y^2)^2} = \frac{4dzd\bar z}{(1-|z|^2)^2}.$$
  Prove that the map $z\mapsto \dfrac{\alpha z+\beta}{\bar\beta z+\bar\alpha}$, where $\begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix}\in SU(1,1)$, is an isometry of (D^2,h).

(3) Let (M,g) be a complete Riemannian manifold of constant sectional curvature K. Suppose that $n=\dim M\ge 2$. Given any $p\in M$, and any orthonormal basis $\{e_1,\dots,e_n\}$ of T_pM, define $\phi:(0,\infty)\times S^{n-1}\to M$ by
$$\phi(r,(x_1,\dots,x_n)) = \exp_p\big(r(\sum_{i=1}^n x_ie_i)\big).$$
Prove that $\phi^*g = dr^2+u_K(r)g_{can}$, where g_{can} is the canonical metric on S^{n-1}, and
$$u_K(r) = \begin{cases} \frac{\sin^2(r\sqrt{K})}{K}, & \text{if } K>0; \\ r^2, & \text{if } K=0; \\ \frac{\sinh^2(r\sqrt{-K})}{-K}, & \text{if } K<0. \end{cases}$$
(Hint: [dC] page 112–113, 2.3 Example.)

(4) [dC] page 181 Exercise 4.