Mathematics G4402. Modern Geometry **Fall 2015**

Assignment 12

Due Wednesday, December 9, 2015

(1) Let $\mathcal{H} = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$ be the upper half plane. Define a Riemannian metric g on $\mathcal{H}$ by

$$g = \frac{dx^2 + dy^2}{y^2}.$$

Prove that $(\mathcal{H}, g)$ has constant sectional curvature -1.

(2) Let G be a Lie group with a bi-invariant metric $\langle\ ,\ \rangle$. Let $X, Y, Z \in \mathcal{X}(G)$ be unit left invariant vector fields on G.

(a) Show that $\nabla_X Y = \frac{1}{2}[X,Y]$. (Hint: see do Carmo page 103, Problem 1(a).)

(b) Conclude from (a) that $R(X,Y)Z = \frac{1}{4}[[X,Y],Z]$.

(c) Prove that if X and Y are orthonormal then the sectional curvature $K(\sigma)$ of G with respect to the plane σ spanned by X and Y are given by

$$K(\sigma) = \frac{1}{4}\|[X,Y]\|.^2$$

(3) (2nd Bianchi identity) Let (M,g) be a Riemannian manifold. Prove that

$$\nabla R(X,Y,Z,W,T) + \nabla R(X,Y,W,T,Z) + \nabla R(X,Y,T,Z,W) = 0$$

for all $X,Y,Z,W,T \in \mathcal{X}(M)$. (Hint: see do Carmo, page 106, Problem 7.)

(4) Let ∇ be an affine connection on a smooth n-manifold M, and let T be an (r,s) tensor on M. Let $(x_1,\ldots,x_n)$ be local coordinates on a coordinate neighborhood U in M. We use the Einstein summation convention. On U, the affine connection ∇ is given by

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \Gamma_{ij}^k \frac{\partial}{\partial x_k}, \quad \Gamma_{ij}^k \in C^\infty(U),$$

and the tensor T can be written as

$$T = T^{i_1\cdots i_r}_{j_1\cdots j_s} \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx_{j_1} \otimes \cdots \otimes dx_{j_s},$$

where $T^{i_1\cdots i_r}_{j_1\cdots j_s} \in C^\infty(U)$. Define $T^{i_1\cdots i_r}_{j_1\ldots j_s,k} \in C^\infty(U)$ by

$$\nabla_{\frac{\partial}{\partial x_k}} T = T^{i_1\cdots i_r}_{j_1\cdots j_s,k} \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx_{j_1} \otimes \cdots \otimes dx_{j_s}.$$

Show that

$$T^{i_1\cdots i_r}_{j_1\cdots j_s,k} = \frac{\partial}{\partial x_k} T^{i_1\ldots i_r}_{j_1\cdots j_s} + \sum_{\alpha=1}^{r} \Gamma^{i_\alpha}_{k\,m} T^{i_1\cdots i_{\alpha-1}\,m\,i_{\alpha+1}\cdots i_r}_{j_1\cdots j_s} - \sum_{\beta=1}^{s} \Gamma^m_{k\,j_\beta} T^{i_1\cdots i_r}_{j_1\cdots j_{\beta-1}\,m\,j_{\beta+1}\cdots j_s}.$$