

Mathematics G4402. Modern Geometry
Assignment 10

Fall 2015

Due Wednesday, November 25, 2015

- (1) Let $F : M \rightarrow N$ be a smooth map between smooth manifolds, and define $F_* : \mathcal{X}(M) \rightarrow C^\infty(M, F^*TN)$ by

$$(F_*X)(p) = dF_p(X(p)) \in T_{F(p)}N = (F^*TN)_p$$

where $X \in \mathcal{X}(M)$ and $p \in M$. Let h be a Riemannian metric on N , let ∇ be the Levi-Civita connection on (N, h) , and let $D = F^*\nabla$ be the pull back connection on F^*TN . Prove the following statements:

- (a) (symmetric) For all $X, Y \in \mathcal{X}(M)$,

$$D_X(F_*Y) - D_Y(F_*X) = F_*([X, Y]).$$

- (b) (compatible with the metric) For all $X \in \mathcal{X}(M)$ and $V, W \in C^\infty(M, F^*TN)$,

$$X\langle V, W \rangle = \langle D_XV, W \rangle + \langle V, D_XW \rangle.$$

- (2) (geodesic frame) Let (M, g) be a Riemannian manifold of dimension n and let $p \in M$. Show that there exists an open neighborhood $U \subset M$ of p and n vector fields $E_1, \dots, E_n \in \mathcal{X}(U)$ such that (i) for all $q \in U$, $\{E_1(q), \dots, E_n(q)\}$ is an orthonormal basis of T_qM , and (ii) $(\nabla_{E_i}E_j)(p) = 0$.
- (3) (normal coordinates) Let (M, g) be a Riemannian manifold of dimension n and let $p \in M$. Show that there exist local coordinates x_1, \dots, x_n on an open neighborhood $U \subset M$ of p such that $g_{ij}(p) = \delta_{ij}$ and $\Gamma_{ij}^k(p) = 0$ for $i, j, k \in \{1, \dots, n\}$, (Hint: see do Carmo page 86, Problem 14.)