Mathematics G4402. Modern Geometry **Fall 2015**

Assignment 9

Due Wednesday, November 18, 2015

- I is an open interval in $\mathbb{R}$, i.e., $I = (a, b)$ for some $-\infty \le a < b \le +\infty$.
- [dC]= do Carmo, *Riemannian Geometry*

(1) Let M be a submanifold of $\mathbb{R}^N$, and let g be the Riemannian metric induced from the Eulidean metric on $\mathbb{R}^N$. Let $\gamma : I \to M$ be a smooth curve in M. Then $\gamma(t) = (x_1(t), \ldots, x_N(t))$ where x_i are smooth functions on I. Define

$$\frac{d^2\gamma}{dt^2} = \sum_{i=1}^{N} \frac{d^2 x_k}{dt^2} \frac{\partial}{\partial x_k}.$$

Prove that

$$\frac{D}{dt}\frac{d\gamma}{dt} = \pi_{\gamma(t)}(\frac{d^2\gamma}{dt^2})$$

where $\frac{D}{dt}$ is the covariant derivative of the Levi-Civita connection on (M, g), and $\pi_{\gamma(t)}$ is the orthogonal projection from $T_{\gamma(t)}\mathbb{R}^N$ to $T_{\gamma(t)}M$. (Hint: You may use Problem (4) of Assignment 8.)

(2) Let $(\vec{a}, \vec{b}) \in TS^n = \{(\vec{x}, \vec{y}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid |\vec{x}| = 1, \vec{x} \cdot \vec{y} = 0\}$. Assume that $\vec{b} \neq \vec{0}$. Prove that

$$\gamma : \mathbb{R} \to S^n, \quad t \mapsto \cos(|\vec{b}|t)\vec{a} + \sin(|\vec{b}|t)\frac{\vec{b}}{|\vec{b}|}$$

is the unique geodesic on (S^n, g_{can}) such that $\gamma(0) = \vec{a}$ and $\gamma'(0) = \vec{b}$. (Hint: You may use Problem (1) above.)

(3) [dC] Chapter 3 Exercise 1 a) and b).

(4) Let X be a left invariant vector field on a Lie group G, and let $\gamma : I \to G$ be an integral curve of X. Prove that γ is a geodesic with respect to any bi-invariant metric on G. (Hint: see [dC] page 80, Exercise 3)