Mathematics G4402. Modern Geometry **Fall 2015**

Assignment 7

Due on Wednesday, November 4, 2015

[dC]= do Carmo, *Riemannian Geometry*

(1) Let G be a Lie group, and let $\mathfrak{g} = T_eG$ be its Lie algebra, where e is the identity element of G. Define $i : G \to G$ by $i(x) = x^{-1}$. Prove the following statements.
   (a) If X is a left invariant vector field on G then i_*X is a right invariant vector field on G.
   (b) For any $\xi \in \mathfrak{g}$, the map $t \mapsto \exp(t\xi)$ is a group homomorphism form $\mathbb{R}$ to G.
   (c) The differential of i at the e is $di_e = -\mathrm{Id}_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}$, where $\mathrm{Id}_{\mathfrak{g}}$ is the identity map from $\mathfrak{g}$ to $\mathfrak{g}$.
   (d) Given any $\xi \in \mathfrak{g}$, let X^L_ξ (resp. X^R_ξ) be the unique left (resp. right) invariant vector field on G such that $X^L_\xi(e) = \xi$ (resp. $X^R_\xi(e) = \xi$). Then for any $\xi, \eta \in \mathfrak{g}$,
$$[X^R_\xi, X^R_\eta](e) = -[X^L_\xi, X^L_\eta](e).$$

(2) Let G_1 and G_2 be Lie groups, and let $e_1 \in G_1$ and $e_2 \in G_2$ be the identity elements. Suppose that $f : G_1 \to G_2$ is a group homomorphism and a smooth map. Prove that $df_{e_1} : T_{e_1}G \to T_{e_2}H$ is a Lie algebra homomorphism.

(3) Let $a_{ij} : GL(n, \mathbb{R}) \to \mathbb{R}$ be the entries of the matrix, so that a_{ij}, $i, j = 1, \ldots, n$ are global coordinates on $GL(n, \mathbb{R})$. Let $\tilde{g}_n$ be the Riemannian metric on $GL(n, \mathbb{R})$ defined by $\tilde{g}_n = \sum_{i,j=1}^n da_{ij}^2$. Let $i : SO(n) \to GL(n, \mathbb{R})$ be the inclusion, which is a smooth embedding. Show that $g_n = i^*\tilde{g}_n$ is a bi-invariant Riemannian metric on $SO(n)$.

(4) Let G be a compact connected Lie group ($\dim G = n$).
   (a) Let ω be a left invariant C^∞ n-form on G, that is, $L_x^*\omega = \omega$ for all $x \in G$. Prove that ω is right invariant. (Hint: see [dC] page 47.)
   (b) Show that there exists a left-invariant C^∞ n-form on G.
   (c) Let $\langle\ ,\ \rangle$ be a left invariant metric on G, and let ω be a left invariant C^∞ n-form on G such that $\int_G \omega > 0$. Define a new Riemannian metric $\langle\langle\ ,\ \rangle\rangle$ on G by
$$\langle\langle u, v\rangle\rangle_y = \int_G \langle (dR_x)_y(u), (dR_x)_y(v)\rangle_{yx}\, \omega, \qquad x, y \in G, \quad u, v \in T_yG.$$
   Prove that the new Riemannian metric $\langle\langle\ ,\ \rangle\rangle$ is bi-invariant.