Mathematics G4402. Modern Geometry **Fall 2015**

Assignment 6

Due on Wednesday, October 21, 2015

(1) Define a smoooth function Q on $\mathbb{R}^{n+1}$ by

$$Q(x_0, x_1, \ldots, x_n) = -x_0^2 + x_1^2 + \cdots + x_n^2.$$

Define a smooth $(0, 2)$ symmetric tensor q on $\mathbb{R}^{n+1}$ by

$$q = -dx_0^2 + dx_1^2 + \cdots + dx_n^2.$$

(a) (Hyperbolic space) Note that -1 is a regular value of the smooth function Q, so

$$H^n = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid Q(x_0, \ldots, x_n) = -1, x_0 > 0\}$$

is an n-dimensional submanifold of $\mathbb{R}^{n+1}$. Let $i : H^n \hookrightarrow \mathbb{R}^{n+1}$ be the inclusion map, and define $g = i^* q \in C^\infty(H^n, S^2 T^* H^n)$. Show that g is positive definite, so it is a Riemannian metric on H^n.

(b) (Poincaré disk) Show that $(x_0, \ldots, x_n) \mapsto \dfrac{1}{x_0 + 1}(x_1, \ldots, x_n)$ defines a diffeomorphism f from H^n onto the unit disk

$$D^n = \{(y_1, \ldots, y_n) \in \mathbb{R}^n \mid y_1^2 + \cdots + y_n^2 < 1\}.$$

Show that $(f^{-1})^* g = \rho \sum_{i=1}^n dy_i^2$ for some smooth, positive function ρ on D^n, and find $\rho(y_1, \ldots, y_n)$.

(c) (Poincaré upper half space) Let $\mathcal{H}^n = \{(y_1, \ldots, y_n) \in \mathbb{R}^n \mid y_n > 0\}$ be the n-dimensional upper half space. Define a Riemannian metric h on $\mathcal{H}_n$ by

$$h = \frac{dy_1^2 + \cdots + dy_n^2}{y_n^2}.$$

Prove that $(\mathcal{H}^n, h)$ is isometric to $(D^n, (f^{-1})^* g)$. (Hint: see page 56-57 of S. Gallot, D. Hulin, and J. Lafontanie's *Riemannian Geometry*, Third Edition.)

(2) Let T^2 be embedded in $\mathbb{R}^3$ as image of $\mathbb{R}^2$ by the map Φ defined by

$$\Phi(\theta, \phi) = ((a + b\cos\theta)\cos\phi, (a + b\cos\theta)\sin\phi, b\sin\theta)$$

where $a > b > 0$. Let g the Riemannian metric induced on T^2 by the Euclidean metric of $\mathbb{R}^3$.

(a) Write g in the form $g = Ed\theta^2 + F(d\theta d\phi + d\phi d\theta) + Gd\phi^2$.

(b) Compute the volume of (T^2, g).

(3) Show that any isometry of the Euclidean space $\mathbb{R}^n$ must take straight lines to straight lines. Show that the only isometries of $\mathbb{R}^n$ are those of the form $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ for constant $A \in O(n)$, $\mathbf{b} \in \mathbb{R}^n$.

(4) Let $G \times M \to M$ be a properly discontinuous action of a group G on a smooth manifold M. Let φ_g, $g \in G$, and the smooth manifold M/G, be defined as in 4.8 Example on page 22 and 23 of do Carmo's *Riemannian Geometry*.

(a) Prove that M/G is orientable if and only if there exists an orientation on M that is preserved by all φ_g, $g \in G$.

(b) Prove that $P^n(\mathbb{R})$ is orientable if and only if n is odd.