

Mathematics G4402. Modern Geometry
Assignment 5

Fall 2015

Due on Wednesday, October 14, 2015

[GHL] = S. Gallot, D. Hulin, J. Lafontaine, *Riemannian Geometry*, Third Edition (Faculty and students at Columbia have access of the full text online.)

- (1) Let X, Y be smooth vector fields on a smooth manifold M . Show that

$$L_X L_Y T - L_Y L_X T = L_{[X, Y]} T$$

for any smooth tensor T on M . (See [GHL] page 39.)

- (2) Let X, Y be smooth vector fields on a smooth n -manifold M , and let $s \in \{0, 1, \dots, n\}$. Prove the following identities.
- (a) $L_X \omega = d(i_X \omega) + i_X (d\omega)$ for any $\omega \in \Omega^s(M)$. (See [GHL] page 43, 44.)
- (b) $L_X (i_Y \omega) - i_Y (L_X \omega) = i_{[X, Y]} \omega$ for any $\omega \in \Omega^s(M)$.
- (c) $L_f X \omega = df \wedge i_X \omega + f L_X \omega$ for any $f \in C^\infty(M)$ and any $\omega \in \Omega^s(M)$.
- (3) Let ω be a smooth s -form on a smooth manifold M . Prove that for any smooth vector fields X_0, X_1, \dots, X_s on M

$$d\omega(X_0, X_1, \dots, X_s) = \sum_{i=0}^s (-1)^i X_i \left(\omega(X_0, \dots, \hat{X}_i, \dots, X_s) \right) \\ + \sum_{0 \leq i < j \leq s} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_s)$$

(See [GHL] page 44.)

- (4) Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Let $i : S^2 \hookrightarrow \mathbb{R}^3$ be the inclusion map, which is a smooth embedding. We define

$$\tilde{\omega} = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy \in \Omega^2(\mathbb{R}^3), \\ \tilde{\eta} = -ydx + xdy \in \Omega^1(\mathbb{R}^3),$$

and let $\omega = i^* \tilde{\omega} \in \Omega^2(S^2)$, $\eta = i^* \tilde{\eta} \in \Omega^1(S^2)$.

- (a) Define two smooth vector fields on \mathbb{R}^3 :

$$\tilde{X} = -zx \frac{\partial}{\partial x} - zy \frac{\partial}{\partial y} + (x^2 + y^2) \frac{\partial}{\partial z}, \quad \tilde{Y} = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}.$$

Show that $\tilde{X}(p), \tilde{Y}(p) \in T_p S^2$ for all $p \in S^2$, so that we may define two smooth vector fields X, Y on S^2 by

$$X(p) = \tilde{X}(p), \quad Y(p) = \tilde{Y}(p), \quad p \in S^2.$$

- (b) Compute $L_X \omega, L_Y \omega, L_X Y, L_X \eta, L_Y \eta$.
- (c) $d\eta = \lambda \omega$ for some $\lambda \in C^\infty(S^2)$. Find $\lambda(x, y, z)$ for all $(x, y, z) \in S^2$.
- (d) $i_X \omega \wedge i_Y \omega = \phi \omega$ for some $\phi \in C^\infty(S^2)$. Find $\phi(x, y, z)$ for all $(x, y, z) \in S^2$.
- (e) Let $\pi : S^2 \rightarrow P_2(\mathbb{R}^2) = S^2 / \{\pm 1\}$ be the projection, which is a surjective local diffeomorphism. Does there exist $\tilde{\omega} \in \Omega^2(P_2(\mathbb{R}))$ such that $\omega = \pi^* \tilde{\omega}$?