

Mathematics G4402. Modern Geometry
Assignment 4

Fall 2015

Due on Wednesday, October 7, 2015

An open interval in \mathbb{R} is of the form (a, b) , where $-\infty \leq a < b \leq +\infty$.

- (1) Let X, Y, Z be smooth vector fields on a smooth manifold. Verify the Jacobi identity:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

- (2) Let X be a smooth vector field on a smooth manifold M , and let $\gamma : I \rightarrow M$ be a nonconstant integral curve of X , where I is an open interval in \mathbb{R} . Prove the following statements.

(a) γ is an immersion.

(b) If γ is not injective, then there exists a smooth embedding $i : S^1 \rightarrow M$ such that $i(S^1) = \gamma(I)$.

- (3) Let X be the vector field on \mathbb{R} defined by $X(x) = x^2 \frac{\partial}{\partial x}$. Given $x \in \mathbb{R}$, let $\phi_x : I_x \rightarrow \mathbb{R}$ be the unique integral curve of X such that $\phi_x(0) = x$, where I_x is an open interval containing 0, and ϕ_x cannot be extended to a larger open interval containing I_x . Find ϕ_x and I_x for all $x \in \mathbb{R}$.

- (4) Let X, Y, Z be the vector fields defined on \mathbb{R}^3 by

$$X = z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z}, \quad Y = x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x}, \quad Z = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}.$$

(a) Show that the map $(a, b, c) \mapsto aX + bY + cZ$ is an isomorphism from \mathbb{R}^3 onto a subspace of the space of smooth vector fields on \mathbb{R}^3 , and that the bracket of vector fields on \mathbb{R}^3 corresponds to the cross product on \mathbb{R}^3 .

(b) Compute the flow of the vector field $aX + bY + cZ$ where $a, b, c \in \mathbb{R}$.