Mathematics G4402. Modern Geometry **Fall 2015**

Assignment 3

Due on Wednesday, September 30, 2015

(1) Let M be a C^k manifold of dimension n, where $k \geq 1$. Let (U, ϕ) be a chart for M around a point $p \in M$, and let $\theta_{(U,\phi,p)} : \mathbb{R}^n \to T_pM$ be defined as in class:

$$\theta_{(U,\phi,p)}(\vec{u}) = [(U, \phi, \vec{u})].$$

Verify that $\theta_{(U,\phi,p)}$ is a bijection.

(2) (universal line bundle) Recall that

$$P_n(\mathbb{R}) = \{\ell \subset \mathbb{R}^{n+1} \mid \ell \text{ is a 1 dimensional linear subspace of } \mathbb{R}^{n+1}\}.$$

Define

$$E = \{(\ell, v) \in P_n(\mathbb{R}) \times \mathbb{R}^{n+1} \mid v \in \ell\} \subset P_n(\mathbb{R}) \times \mathbb{R}^{n+1}.$$

Let $p_1 : P_n(\mathbb{R}) \times \mathbb{R}^{n+1} \to P_n(\mathbb{R})$ be the projection to the first factor, and let $\pi : E \to P_n(\mathbb{R})$ be the restriction of p_1 to E. Prove that $\pi : E \to P_n(\mathbb{R})$ is a C^∞ vector bundle of rank 1 over $P_n(\mathbb{R})$.

(3) Show that if δ is a derivation on $C^0_0(\mathbb{R}^n)$ then $\delta = 0$.

(4) Let M be a smooth submanifold of a smooth manifold N, and let X, Y be smooth vector fields on M. Let $p \in M$ and let U be an open neighborhood of p in N.

(a) Suppose that $\tilde{X}, \tilde{Y} \in C^\infty(U, TU)$ are smooth vector fields on U such that for all $q \in U \cap M$

$$\tilde{X}(q) = X(q) \in T_qM, \quad \tilde{Y}(q) = Y(q) \in T_qM.$$

Show that $[\tilde{X}, \tilde{Y}](q) \in T_qM$ for all $q \in U \cap M$.

(b) Let f be a smooth function on M, and let $\tilde{f}$ be a smooth function on U such that $\tilde{f}(q) = f(q)$ for all $q \in U \cap M$. Let $g = [X, Y]f \in C^\infty(M)$ and let $\tilde{g} = [\tilde{X}, \tilde{Y}]\tilde{f} \in C^\infty(U)$. Show that $\tilde{g}(q) = g(q)$ for all $q \in U \cap M$.