Mathematics G4402. Modern Geometry **Fall 2015**

Assignment 2

Due on Wednesday, September 23, 2015

(1) Let M be a subset of $\mathbb{R}^n$. Show that M is an m-dimensional C^∞ submanifold of $\mathbb{R}^n$ (where $m < n$) if and only if for any point $p \in M$, there exists a subset $I = \{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$ and an open neighborhood U of p in M, such that $\phi : U \to \mathbb{R}^m$ given by $(x_1, \ldots, x_n) \mapsto (x_{i_1}, \ldots, x_{i_m})$ maps U homeomophically to an open subset $\Omega \subset \mathbb{R}^m$, and U is the graph of a C^∞ map $F : \Omega \to \mathbb{R}^J$, where $J = \{1, \ldots, n\} \setminus I$. In other words, there are C^∞ functions $f_j : \Omega \to \mathbb{R}$ for $j \in J$, such that

$$U = \{(x_1, \ldots, x_n) : (x_{i_1}, \ldots, x_{i_m}) \in \Omega,\ x_j = f_j(x_{i_1}, \ldots, x_{i_m}) \text{ if } j \in J\}.$$

(2) Prove that the tangent bundle TM of a smooth manifold M is an orientable manifold (even though M may not be).

(3) Let $p(x_1, \ldots, x_k) \in \mathbb{R}[x_1, \ldots, x_k]$ be a homogeneous polynomial of degree m, i.e.,

$$p(tx_1, \ldots, tx_k) = t^m p(x_1, \ldots, x_k).$$

We assume that $m \geq 2$.

(a) Prove that if $a \neq 0$ then

$$X_a = \{x \in \mathbb{R}^k \mid p(x) = a\}$$

is a $k - 1$ dimensional submanifold of $\mathbb{R}^k$. [Hint: Use Euler's identity for homogeneous polynomials

$$\sum_{i=1}^{k} x_i \frac{\partial p}{\partial x_i} = m \cdot p$$

to prove that 0 is the only critical value of p.]

(b) Prove that X_a is diffeomorphic to X_1 if $a > 0$, and X_a is diffeomorphic to X_{-1} if $a < 0$.

(4) Let $M_n(\mathbb{R})$ be the space of $n \times n$ matrices (with real entries). We assume that $n \geq 2$. Let $SL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det A = 1\}$.

(a) Show that $SL(n, \mathbb{R})$ is an $(n^2 - 1)$-dimensional submanifold of $M_n(\mathbb{R})$.

(b) Describe $T_{I_n} SL(n, \mathbb{R})$ (the tangent space to $SL(n, \mathbb{R})$ at the identity matrix I_n) explicitly as a linear subspace of $M_n(\mathbb{R})$.

(c) Describe $TSL(n, \mathbb{R})$ (the tangent bundle of $SL(n, \mathbb{R})$) explicitly as a subset of $M_n(\mathbb{R}) \times M_n(\mathbb{R})$.