

Assignment 11

Due Monday, December 12, 2011

- (1) Let (M, g) be a Riemannian manifold of dimension $n \geq 2$. Show that in the normal coordinate system at $p \in M$, $\det(g_{ij})$ has the following expansion:

$$\det(g_{ij})(x) = 1 - \frac{1}{3} \sum_{ij} R_{ij}(p)x_i x_j - \frac{1}{6} \sum_{ijk} R_{ij,k}(p)x_i x_j x_k + \sum_{ijkl} \left(-\frac{1}{20} R_{ij,kl}(p) - \frac{1}{90} \sum_{hlm} R_{hijm}(p) R_{hklm}(p) + \frac{1}{18} R_{ij}(p) R_{kl}(p) \right) x_i x_j x_k x_l + O(|x|^5),$$

where $|x| = \sqrt{\sum_{i=1}^n x_i^2}$. [Hint: you may use the Taylor expansion of g_{ij} derived in class.]

- (2) do Carmo page 122 Exercise 6.
 (3) do Carmo page 122 Exercise 7.
 (4) Let (\bar{M}, \bar{g}) be a Riemannian manifold of dimension $n + 1$. Let $f : M \rightarrow \mathbb{R}$ be a smooth function, and let $a \in \mathbb{R}$ be a regular value of f such that $M = f^{-1}(a)$ is nonempty. Then M is an n -dimensional submanifold of \bar{M} . Let $i : M \rightarrow \bar{M}$ be the inclusion map, and let $g = i^* \bar{g}$, so that $i : (M, g) \rightarrow (\bar{M}, \bar{g})$ is an isometric embedding. Let $\text{Hess} f = \bar{\nabla} df$ be the Hessian of f , where $\bar{\nabla}$ is the Levi-Civita connection on (\bar{M}, \bar{g}) . Prove the following statements.

(a) For any $p \in M$ and $x, y \in T_p M$, we have

$$\text{grad} f(p) \in (T_p M)^\perp, \quad H_{\text{grad} f(p)}(x, y) = -\text{Hess} f(p)(x, y).$$

(b) Let H be the mean curvature of M in \bar{M} with respect to the unit normal $\frac{\text{grad} f}{|\text{grad} f|}$. Then

$$H = -\frac{1}{n} \text{div} \left(\frac{\text{grad} f}{|\text{grad} f|} \right)$$

where the gradient and the divergence are defined by \bar{g} .