## Mathematics G4402. Modern Geometry  Fall 2011

## Assignment 10

*Due Monday, December 5, 2011*

(1) (2nd Bianchi identity) Let $(M, g)$ be a Riemannian manifold. Prove that

$$\nabla R(X, Y, Z, W, T) + \nabla R(X, Y, W, T, Z) + \nabla R(X, Y, T, Z, W) = 0$$

for all $X, Y, Z, W, T \in \mathcal{X}(M)$. (Hint: see do Carmo, page 106, Problem 7.)

(2) (Einstein manifolds) A Riemannian manifold $(M, g)$ is called an *Einstein manifold* if $\mathrm{Ric} = \lambda g$ for some $\lambda \in C^\infty(M)$.

Let $(M, g)$ be a connected Einstein manifold with $\mathrm{Ric} = \lambda g$. Prove that:

(a) If $\dim M \geq 3$ then $\lambda$ is a constant. (Hint: see do Carmo page 108, Problem 10.)

(b) If $\dim M = 3$ then $M$ has constant sectional curvature.

(3) Let $g$ be a Riemannian metric on a manifold $M$, and let $\tilde{g} = r^2 g$, where $r > 0$ is a constant. Let $R$, $K$, and $S$ be the curvature tensor, sectional curvature, and scalar curvature of $(M, g)$, respectively; let $\tilde{R}$, $\tilde{K}$, and $\tilde{S}$ be the curvature tensor, sectional curvature, and scalar curvature of $(M, \tilde{g})$, respectively. Prove that

$$\tilde{R}^l_{ijk} = R^k_{ijk}, \quad \tilde{R}_{ijkl} = r^2 R_{ijkl},$$
$$\tilde{K} = r^{-2} K, \quad \tilde{R}_{ij} = R_{ij}, \quad \tilde{S} = r^{-2} S$$

(4) Let $\nabla$ be an affine connection on a smooth $n$-manifold $M$, and let $T$ be an $(r, s)$ tensor on $M$. Let $(x_1, \ldots, x_n)$ be local coordinates on a coordinate neighborhood $U$ in $M$. We use the Einstein summation convention. On $U$, the affine connection $\nabla$ is given by

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \Gamma^k_{ij} \frac{\partial}{\partial x_k}, \quad \Gamma^k_{ij} \in C^\infty(U),$$

and the tensor $T$ can be written as

$$T = T^{i_1 \cdots i_r}_{j_1 \cdots j_s} \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx_{j_1} \otimes \cdots \otimes dx_{j_s},$$

where $T^{i_1 \cdots i_r}_{j_1 \cdots j_s} \in C^\infty(U)$. Define $T^{i_1 \cdots i_r}_{j_1 \ldots j_s, k} \in C^\infty(U)$ by

$$\nabla_{\frac{\partial}{\partial x_k}} T = T^{i_1 \cdots i_r}_{j_1 \cdots j_s, k} \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx_{j_1} \otimes \cdots \otimes dx_{j_s}.$$

Show that

$$T^{i_1 \cdots i_r}_{j_1 \cdots j_s, k} = \frac{\partial}{\partial x_k} T^{i_1 \ldots i_r}_{j_1 \cdots j_s} + \sum_{\alpha=1}^{r} \Gamma^{i_\alpha}_{k\, m} T^{i_1 \cdots i_{\alpha-1}\, m\, i_{\alpha+1} \cdots i_r}_{j_1 \cdots j_s} - \sum_{\beta=1}^{s} \Gamma^{m}_{k\, j_\beta} T^{i_1 \cdots i_r}_{j_1 \cdots j_{\beta-1}\, m\, j_{\beta+1} \cdots j_s}.$$