

Mathematics G4402. Modern Geometry
Assignment 9

Fall 2011

Due Monday, November 28, 2011

- (1) Let M be a submanifold of \mathbb{R}^N , and let g be the Riemannian metric induced from the Euclidean metric on \mathbb{R}^N . Let $\gamma : (a, b) \rightarrow M$ be a smooth curve in M . Then $\gamma(t) = (x_1(t), \dots, x_N(t))$ where x_i are smooth functions on I . Define

$$\frac{d^2\gamma}{dt^2} = \sum_{k=1}^N \frac{d^2x_k}{dt^2} \frac{\partial}{\partial x_k}.$$

Prove that

$$\frac{D}{dt} \frac{d\gamma}{dt} = \pi_{\gamma(t)} \left(\frac{d^2\gamma}{dt^2} \right)$$

where $\frac{D}{dt}$ is the covariant derivative of the Levi-Civita connection on (M, g) , and $\pi_{\gamma(t)}$ is the orthogonal projection from $T_{\gamma(t)}\mathbb{R}^N$ to $T_{\gamma(t)}M$. (Hint: You may use Problem 3 of Assignment 7.)

- (2) Let $(\vec{a}, \vec{b}) \in TS^n = \{(\vec{x}, \vec{y}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid |\vec{x}| = 1, \vec{x} \cdot \vec{y} = 0\}$. Assume that $\vec{b} \neq \vec{0}$. Use Problem 1 to prove that

$$\gamma : \mathbb{R} \rightarrow S^n, \quad t \mapsto \cos(|\vec{b}|t)\vec{a} + \sin(|\vec{b}|t)\frac{\vec{b}}{|\vec{b}|}$$

is the unique geodesic on (S^n, g_{can}) such that $\gamma(0) = \vec{a}$ and $\gamma'(0) = \vec{b}$.

- (3) Let G be a Lie group with a bi-invariant metric $\langle \cdot, \cdot \rangle$. Let $X, Y, Z \in \mathcal{X}(G)$ be unit left invariant vector fields on G .
- (a) Show that $\nabla_X Y = \frac{1}{2}[X, Y]$. (Hint: see do Carmo page 103, Problem 1(a).)
- (b) Conclude from (a) that $R(X, Y)Z = \frac{1}{4}[[X, Y], Z]$.
- (c) Prove that if X and Y are orthonormal then the sectional curvature $K(\sigma)$ of G with respect to the plane σ spanned by X and Y are given by

$$K(\sigma) = \frac{1}{4} \| [X, Y] \|^2.$$

- (4) Let $\mathcal{H} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ be the upper half plane. Define a Riemannian metric g on \mathcal{H} by

$$g = \frac{dx^2 + dy^2}{y^2}.$$

Prove that (\mathcal{H}, g) has constant sectional curvature -1 .