Mathematics G4402. Modern Geometry **Fall 2011**

Assignment 8

Due Monday, November 21, 2011

- I is an open interval in $\mathbb{R}$, i.e., $I = (a, b)$ for some $-\infty \leq a < b \leq +\infty$.
- [dC]= do Carmo, *Riemannian Geometry*

(1) Let X be a left invariant vector field on a Lie group G, and let $\gamma : I \to G$ be an integral curve of X. Prove that γ is a geodesic with respect to any bi-invariant metric on G. (Hint: see [dC] page 80, Problem 3)

(2) Let $F : M \to N$ be a smooth map between smooth manifolds, and define $F_* : \mathcal{X}(M) \to C^\infty(M, F^*TN)$ by

$$(F_*X)(p) = dF_p(X(p)) \in T_{F(p)}N = (F^*TN)_p$$

where $X \in \mathcal{X}(M)$ and $p \in M$. Let h be a Riemannian metric on N, let ∇ be the Levi-Civita connection on (N, h), and let $D = F^*\nabla$ be the pull back connection on F^*TN. Prove the following statements:

(a) (symmetric) For all $X, Y \in \mathcal{X}(M)$,

$$D_X(F_*Y) - D_Y(F_*X) = F_*([X, Y]).$$

(b) (compatible with the metric) For all $X \in \mathcal{X}(M)$ and $V, W \in C^\infty(M, F^*TN)$,

$$X\langle V, W\rangle = \langle D_X V, W\rangle + \langle V, D_X W\rangle.$$

(3) (geodesic frame) Let (M, g) be a Riemannian manifold of dimension n and let $p \in M$. Show that there exists an open neighborhood $U \subset M$ of p and n vector fields $E_1, \ldots, E_n \in \mathcal{X}(U)$ such that (i) for all $q \in U$, $\{E_1(q), \ldots, E_n(q)\}$ is an orthonormal basis of T_qM, and (ii) $(\nabla_{E_i} E_j)(p) = 0$.

(4) (normal coordinates) Let (M, g) be a Riemannian manifold of dimension n and let $p \in M$. Show that there exist local coordinates $x_1, \ldots, x_n$ on an open neighborhood $U \subset M$ of p such that $g_{ij}(p) = \delta_{ij}$ and $\Gamma^k_{ij}(p) = 0$ for $i, j, k \in \{1, \ldots, n\}$, (Hint: see do Carmo page 86, Problem 14.)