

Assignment 7

Due November 14, 2011

- (1) Let  $H$  be a closed Lie subgroup of a Lie group  $G$ , and let  $G/H = \{aH \mid a \in G\}$  be the set of *left* cosets of  $H$  in  $G$ . (In other words, let  $H$  act on  $G$  by *right* multiplication and let  $G/H$  be the quotient.)  $G$  acts on  $G/H$  on the *left* by  $G \times G/H \rightarrow G/H$ ,  $(a, bH) \mapsto abH$ . Let  $g$  be a *right* invariant Riemannian metric on  $G$ . By the theorems stated in class, there is a unique Riemannian metric  $\hat{g}$  on  $G/H$  such that  $\pi : (G, g) \rightarrow (G/H, \hat{g})$  is a Riemannian submersion. Prove that if  $g$  is left invariant then  $G$  acts isometrically on  $(G/H, \hat{g})$ .
- (2) Let  $g_n$  be the bi-invariant metric on  $SO(n)$  defined in Problem 3 of Assignment 6. We have seen in class that there is a diffeomorphism

$$f : S^n \rightarrow SO(n+1)/SO(n).$$

Let  $\hat{g}$  be the unique Riemannian metric on  $SO(n+1)/SO(n)$  such that

$$\pi : (SO(n+1), g_{n+1}) \rightarrow (SO(n+1)/SO(n), \hat{g})$$

is a Riemannian submersion. Prove that  $f^*\hat{g} = \lambda g_{\text{can}}$  for some  $\lambda > 0$ , and find  $\lambda$ . (Hint: What is the horizontal space  $H_{I_{n+1}} \subset T_{I_{n+1}}SO(n+1)$ ?)

- (3) Let  $F : (M, g) \rightarrow (N, h)$  be an isometric immersion. For any  $p \in M$ , let  $\pi_p$  be the orthogonal projection from  $T_{F(p)}N$  to the image of  $dF_p : T_pM \rightarrow T_{F(p)}N$ . Let  $X, Y$  be  $C^\infty$  vector fields on  $M$  which are  $F$ -related to  $C^\infty$  vector fields  $\tilde{X}, \tilde{Y}$  on  $N$ , respectively. Let  $\nabla$  and  $\tilde{\nabla}$  be the Levi-Civita connections on  $(M, g)$  and on  $(N, h)$ , respectively. Prove that for any  $p \in M$ ,

$$dF_p((\nabla_X Y)(p)) = \pi_p((\tilde{\nabla}_{\tilde{X}} \tilde{Y})(F(p)))$$

- (4) Let  $\mathcal{H} = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_2 > 0\}$  be the upper half plane, and define a Riemannian metric on  $\mathcal{H}$  by

$$g = \frac{dy_1^2 + dy_2^2}{y_2^2}.$$

- (a) Compute the Christoffel symbols  $\Gamma_{ij}^k$ ,  $i, j, k \in \{1, 2\}$ , for the Levi-Civita connection  $\nabla$  on  $(\mathcal{H}, g)$ .
- (b) Define  $\gamma : \mathbb{R} \rightarrow \mathcal{H}$  by  $\gamma(t) = (t, 1)$ . Then  $\gamma$  is a smooth curve in  $\mathcal{H}$ . Let

$$V(t) = a(t) \frac{\partial}{\partial y_1} + b(t) \frac{\partial}{\partial y_2}$$

be the unique parallel (w.r.t.  $\nabla$ ) vector field along  $\gamma$  such that  $V(0) = \frac{\partial}{\partial y_2}$ . Find  $a(t), b(t)$  for  $t \in \mathbb{R}$ .