Mathematics G4402. Modern Geometry **Fall 2011**

Assignment 6

Due on Monday, October 31, 2011

[dC]= do Carmo, *Riemannian Geometry*
[GHL]= Gallot, Hulin, Lafontaine, *Riemannian Geometry*

(1) Let $G \times M \to M$ be a properly discontinuous action of a group G on a smooth manifold M. Let φ_g, $g \in G$, and the smooth manifold M/G, be defined as in [dC] page 22 and 23, 4.8 Example.
   (a) Prove that M/G is orientable if and only if there exists an orientation on M that is preserved by all φ_g, $g \in G$.
   (b) Prove that $P^n(\mathbb{R})$ is orientable if and only if n is odd.

(2) Let G_1 and G_2 be Lie groups, and let $e_1 \in G_1$ and $e_2 \in G_2$ be the identity elements. Suppose that $f : G_1 \to G_2$ is a group homomorphism and a smooth map. Prove that $df_{e_1} : T_{e_1}G \to T_{e_2}H$ is a Lie algebra homomorphism.

(3) Let $a_{ij} : GL(n, \mathbb{R}) \to \mathbb{R}$ be the entries of the matrix, so that a_{ij}, $i, j = 1, \ldots, n$ are global coordinates on $GL(n, \mathbb{R})$. Let $\tilde{g}_n$ be the Riemannian metric on $GL(n, \mathbb{R})$ defined by $\tilde{g}_n = \sum_{i,j=1}^n da_{ij}^2$. Let $i : SO(n) \to GL(n, \mathbb{R})$ be the inclusion, which is a smooth embedding. Show that $g_n = i^*\tilde{g}_n$ is a bi-invariant Riemannian metric on $SO(n)$.

(4) Let G be a compact connected Lie group ($\dim G = n$).
   (a) Let ω be a left invariant C^∞ n-form on G, that is, $L_x^*\omega = \omega$ for all $x \in G$. Prove that ω is right invariant. (Hint: see [dC] page 47.)
   (b) Show that there exists a left-invariant C^∞ n-form on G.
   (c) Let $\langle\ ,\ \rangle$ be a left invariant metric on G, and let ω be a left invariant C^∞ n-form on G such that $\int_G \omega > 0$. Define a new Riemannian metric $\langle\langle\ ,\ \rangle\rangle$ on G by

$$\langle\langle u, v\rangle\rangle_y = \int_G \langle (dR_x)_y(u), (dR_x)_y(v)\rangle_{yx}\, \omega, \qquad x, y \in G, \quad u, v \in T_yG.$$

   Prove that the new Riemannian metric $\langle\langle\ ,\ \rangle\rangle$ is bi-invariant.