Mathematics G4402. Modern Geometry **Fall 2011**

Assignment 4

Due on Monday, October 10, 2011

[GHL] = S. Gallot, D. Hulin, J. Lafontaine, *Riemannian Geometry*, Third Edition

(1) Let X, Y be smooth vector fields on a smooth manifold M. Show that

$$L_X L_Y T - L_Y L_X T = L_{[X,Y]} T$$

for any smooth tensor T on M. (See [GHL] page 39.)

(2) Let X, Y be smooth vector fields on a smooth n-manifold M, and let $s \in \{0, 1, \dots, n\}$. Prove the following identities.

(a) $L_X\omega = d(i_X\omega) + i_X(d\omega)$ for any $\omega \in \Omega^s(M)$. (See [GHL] page 43, 44.)

(b) $L_X(i_Y\omega) - i_Y(L_X\omega) = i_{[X,Y]}\omega$ for any $\omega \in \Omega^s(M)$.

(c) $L_{fX}\omega = df \wedge i_X\omega + fL_X\omega$ for any $f \in C^\infty(M)$ and any $\omega \in \Omega^s(M)$.

(3) Let ω be a smooth s-form on a smooth manifold M. Prove that for any smooth vector fields $X_0, X_1, \dots, X_s$ on M

$$d\omega(X_0, X_1, \dots, X_s) = \sum_{i=0}^{s} (-1)^i X_i \left(\omega(X_0, \dots, \hat{X}_i, \dots, X_s)\right)$$
$$+ \sum_{0 \le i < j \le s} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_s)$$

(See [GHL] page 44.)

(4) Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Let $i : S^2 \hookrightarrow \mathbb{R}^3$ be the inclusion map, which is a smooth embedding. We define

$$\tilde{\omega} = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy \in \Omega^2(\mathbb{R}^3),$$
$$\tilde{\eta} = -ydx + xdy \in \Omega^1(\mathbb{R}^3),$$

and let $\omega = i^*\tilde{\omega} \in \Omega^2(S^2)$, $\eta = i^*\tilde{\eta} \in \Omega^1(S^2)$.

(a) Define two smooth vector fields on $\mathbb{R}^3$:

$$\tilde{X} = -zx\frac{\partial}{\partial x} - zy\frac{\partial}{\partial y} + (x^2 + y^2)\frac{\partial}{\partial z}, \quad \tilde{Y} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Show that $\tilde{X}(p), \tilde{Y}(p) \in T_pS^2$ for all $p \in S^2$, so that we may define two smooth vector fields X, Y on S^2 by

$$X(p) = \tilde{X}(p), \quad Y(p) = \tilde{Y}(p), \quad p \in S^2.$$

(b) Compute $L_X\omega$, $L_Y\omega$, L_XY, $L_X\eta$, $L_Y\eta$.

(c) $d\eta = \lambda\omega$ for some $\lambda \in C^\infty(S^2)$. Find $\lambda(x, y, z)$ for all $(x, y, z) \in S^2$.

(d) $i_X\omega \wedge i_Y\omega = \phi\omega$ for some $\phi \in C^\infty(S^2)$. Find $\phi(x, y, z)$ for all $(x, y, z) \in S^2$.

(e) Let $\pi : S^2 \to P_2(\mathbb{R}^2) = S^2/\{\pm 1\}$ be the projection, which is a surjective local diffeomorphism. Does there exist $\bar{\omega} \in \Omega^2(P_2(\mathbb{R}))$ such that $\omega = \pi^*\bar{\omega}$?