Mathematics G4402. Modern Geometry **Fall 2011**

Assignment 3

Due on Monday, October 3, 2011

An open interval in $\mathbb{R}$ is of the form (a, b), where $-\infty \le a < b \le +\infty$.

(1) Let M be a smooth submanifold of a smooth manifold N, and let X, Y be smooth vector fields on M. Let $p \in M$ and let U be an open neighborhood of p in N.

(a) Suppose that $\tilde{X}, \tilde{Y} \in C^\infty(U, TU)$ are smooth vector fields on U such that for all $q \in U \cap M$

$$\tilde{X}(q) = X(q) \in T_qM, \quad \tilde{Y}(q) = Y(q) \in T_qM.$$

Show that $[\tilde{X}, \tilde{Y}](q) \in T_qM$ for all $q \in U \cap M$.

(b) Let f be a smooth function on M, and let $\tilde{f}$ be a smooth function on U such that $\tilde{f}(q) = f(q)$ for all $q \in U \cap M$. Let $g = [X, Y]f \in C^\infty(M)$ and let $\tilde{g} = [\tilde{X}, \tilde{Y}]\tilde{f} \in C^\infty(U)$. Show that $\tilde{g}(q) = g(q)$ for all $q \in U \cap M$.

(2) Let X be a smooth vector field on a smooth manifold M, and let $\gamma : I \to M$ be a nonconstant integral curve of X, where I is an open interval in $\mathbb{R}$. Prove the following statements.

(a) γ is an immersion.

(b) If γ is not injective, then there exists a smooth embedding $i : S^1 \to M$ such that $i(S^1) = \gamma(I)$.

(3) Let X be the vector field on $\mathbb{R}$ defined by $X(x) = x^2 \frac{\partial}{\partial x}$. Given $x \in \mathbb{R}$, let $\phi_x : I_x \to \mathbb{R}$ be the unique integral curve of X such that $\phi_x(0) = x$, where I_x is an open interval containing 0, and ϕ_x cannot be extended to a larger open interval containing I_x. Find ϕ_x and I_x for all $x \in \mathbb{R}$.

(4) Let X, Y, Z be the vector fields defined on $\mathbb{R}^3$ by

$$X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad Y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad Z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

(a) Show that the map $(a, b, c) \mapsto aX + bY + cZ$ is an isomorphism from $\mathbb{R}^3$ onto a subspace of the space of smooth vector fields on $\mathbb{R}^3$, and that the bracket of vector fields on $\mathbb{R}^3$ corresponds to the cross product on $\mathbb{R}^3$.

(b) Compute the flow of the vector field $aX + bY + cZ$ where $a, b, c \in \mathbb{R}$.