Mathematics G4402. Modern Geometry **Fall 2011**

Assignment 2

Due on Monday, September 26, 2011

(1) Let $p(x_1, \dots, x_k) \in \mathbb{R}[x_1, \dots, x_k]$ be a homogeneous polynomial of degree m, i.e.,

$$p(tx_1, \dots, tx_k) = t^m p(x_1, \dots, x_k).$$

We assume that $m \geq 2$.

(a) Prove that if $a \neq 0$ then

$$X_a = \{x \in \mathbb{R}^k \mid p(x) = a\}$$

is a $k-1$ dimensional submanifold of $\mathbb{R}^k$. [Hint: Use Euler's identity for homogeneous polynomials

$$\sum_{i=1}^{k} x_i \frac{\partial p}{\partial x_i} = m \cdot p$$

to prove that 0 is the only critical value of p.]

(b) Prove that X_a is diffeomorphic to X_1 if $a > 0$, and X_a is diffeomorphic to X_{-1} if $a < 0$.

(2) Let $M_n(\mathbb{R})$ be the space of $n \times n$ matrices (with real entries). We assume that $n \geq 2$. Let $SL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det A = 1\}$.

(a) Show that $SL(n, \mathbb{R})$ is an $(n^2 - 1)$-dimensional submanifold of $M_n(\mathbb{R})$.

(b) Describe $T_{I_n} SL(n, \mathbb{R})$ (the tangent space to $SL(n, \mathbb{R})$ at the identity matrix I_n) explicitly as a linear subspace of $M_n(\mathbb{R})$.

(c) Describe $TSL(n, \mathbb{R})$ (the tangent bundle of $SL(n, \mathbb{R})$) explicitly as a subset of $M_n(\mathbb{R}) \times M_n(\mathbb{R})$.

(3) (universal line bundle) Recall that

$$P_n(\mathbb{R}) = \{\ell \subset \mathbb{R}^{n+1} \mid \ell \text{ is a 1 dimensional linear subspace of } \mathbb{R}^{n+1}\}.$$

Define

$$E = \{(\ell, v) \in P_n(\mathbb{R}) \times \mathbb{R}^{n+1} \mid v \in \ell\} \subset P_n(\mathbb{R}) \times \mathbb{R}^{n+1}.$$

Let $p_1 : P_n(\mathbb{R}) \times \mathbb{R}^{n+1} \to P_n(\mathbb{R})$ be the projection to the first factor, and let $\pi : E \to P_n(\mathbb{R})$ be the restriction of p_1 to E. Prove that $\pi : E \to P_n(\mathbb{R})$ is a C^∞ vector bundle of rank 1 over $P_n(\mathbb{R})$.