

Assignment 1

Due on Monday, September 19, 2011

In this assignment:

- smooth structure =  $C^\infty$  differentiable structure
- smooth map =  $C^\infty$  map

(1) (Stereographic projection) Let

$$S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$$

be equipped with the subset topology. Let  $N = (0, \dots, 0, 1)$  and  $S = (0, \dots, 0, -1)$  be the north and south poles of  $S^n$ , respectively. Define  $\pi_1 : S^n - \{N\} \rightarrow \mathbb{R}^n$  (resp.  $\pi_2 : S^n - \{S\} \rightarrow \mathbb{R}^n$ ) such that  $(\pi_1(p), 0)$  (resp.  $(\pi_2(p), 0)$ ) is the point at which the line through  $N$  (resp.  $S$ ) and  $p$  intersects the hyperplane  $\{x_{n+1} = 0\}$ . (See do Carmo page 19 Example 4.6.)

(a) Prove that  $\Phi = \{(S^n - \{N\}, \pi_1), (S^n - \{S\}, \pi_2)\}$  is a  $C^\infty$  atlas on  $S^n$ .

(b) Prove that the inclusion map  $i : S^n \rightarrow \mathbb{R}^{n+1}$  is a smooth embedding, where  $S^n$  is equipped with the smooth structure defined by the  $C^\infty$  atlas  $\Phi$ . Therefore the smooth structure defined by  $\Phi$  coincides with the smooth structure on  $S^n$  as a submanifold of  $\mathbb{R}^{n+1}$ .

(2) (Embedding of  $P^2(\mathbb{R})$  in  $\mathbb{R}^4$ ) Let  $F : \mathbb{R}^3 \rightarrow \mathbb{R}^4$  be given by

$$F(x, y, z) = (x^2 - y^2, xy, zx, yz).$$

Let  $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ . Observe that  $f = F|_{S^2}$  satisfies  $f(x, y, z) = f(-x, -y, -z)$ , so that it descends to a map  $\tilde{f} : P^2(\mathbb{R}) = S^2/\{\pm 1\} \rightarrow \mathbb{R}^4$ . Prove that  $\tilde{f}$  is a smooth embedding. [Hint: A bijective continuous map from a compact topological space to a Hausdorff topological space is a homeomorphism.]

(3) Let  $(x, y, z)$  be coordinates on  $\mathbb{R}^3$ . Let  $Y_r$  be the set of points in  $\mathbb{R}^3$  at distance  $r > 0$  from the circle

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, z = 0\}.$$

(a) Let  $A = \{r \in (0, \infty) \mid Y_r \text{ is a smooth submanifold of } \mathbb{R}^3\}$ . Find  $A$ .

(b) Let  $S^1$  be equipped with the smooth structure in (1), and let  $S^1 \times S^1$  be the product manifold (see do Carmo page 31-32 Exercise 1). Prove that  $Y_r$  is  $C^\infty$  diffeomorphic to  $S^1 \times S^1$  for any  $r \in A$ .

(4) Prove that the tangent bundle  $TM$  of a smooth manifold  $M$  is an orientable manifold (even though  $M$  may not be).