

Assignment 12

Due Monday, December 14, 2009

- (1) Let (M, g) be a Riemannian manifold of dimension n , and let $f \in C^\infty(M)$. Let $M \times \mathbb{R}$ be equipped with the product metric \bar{g} . In local coordinates (x_1, \dots, x_n) on M ,

$$g = g_{ij}dx_i dx_j, \quad \bar{g} = g_{ij}dx_i dx_j + dt^2,$$

$$df = f_i dx_i, \quad \text{Hess}(f) = f_{ij}dx_i dx_j$$

where the Hessian is defined by the metric g . The map $\phi_f : M \rightarrow M \times \mathbb{R}$ defined by $x \mapsto (x, f(x))$ is a smooth embedding, so the image $\Gamma_f = \phi_f(M)$ is an n -dimensional submanifold of $M \times \mathbb{R}$.

- (a) Let $\tilde{g} = \phi_f^* \bar{g}$, and write $\tilde{g} = \tilde{g}_{ij}dx_i dx_j$. Show that

$$\tilde{g}_{ij} = g_{ij} + f_i f_j, \quad \tilde{g}^{ij} = g^{ij} - \frac{f^i f^j}{1 + |df|^2}$$

where $f^i = g^{ij} f_j$ and $|df|^2 = g^{ij} f_i f_j$.

- (b) Let H be the mean curvature of $\Gamma_f \subset M \times \mathbb{R}$ with respect to the upward unit normal. Show that

$$n\phi_f^* H = \frac{\tilde{g}^{ij} f_{ij}}{\sqrt{1 + |df|^2}}$$

- (2) do Carmo page 119-120 Exercise 3.
 (3) do Carmo page 122 Exercise 6.
 (4) do Carmo page 122 Exercise 7.