Mathematics G4402. Modern Geometry **Fall 2009**

Assignment 11

Due Monday, December 7, 2009

(1) Let g be a Riemannian metric on a manifold M, and let $\tilde{g} = e^{2f}g$, where f is a smooth function on M. Let ∇ and $\tilde{\nabla}$ be the Levi-Civita connections on (M, g) and $(M, \tilde{g})$, respectively. Prove that for any $X, Y \in \mathcal{X}(M)$,

$$\tilde{\nabla}_X Y = \nabla_X Y + X(f)Y + Y(f)X - g(X, Y)\text{grad}f$$

where $\text{grad}f$ is defined by g.

(2) Let $(\bar{M}, \bar{g})$ be a Riemannian manifold of dimension $n+1$. Let $f : M \to \mathbb{R}$ be a smooth function, and let $a \in \mathbb{R}$ be a regular value of f such that $M = f^{-1}(a)$ is nonempty. Then M is an n-dimensional submanfold of M. Let $i : M \to \bar{M}$ be the inclusion map, and let $g = i^*\bar{g}$, so that $i : (M, g) \to (\bar{M}, \bar{g})$ is an isometric embedding. Let $\text{Hess}f = \bar{\nabla} df$ be the Hessian of f, where $\bar{\nabla}$ is the Levi-Civita connection on $(\bar{M}, \bar{g})$. Prove the following statements.

(a) For any $p \in M$ and $x, y \in T_pM$, we have

$$\text{grad}f(p) \in (T_pM)^{\perp}, \qquad H_{\text{grad}f(p)}(x, y) = -\text{Hess}f(p)(x, y).$$

(b) Let H be the mean curvature vector of M in $\bar{M}$ with respect to the unit normal $\frac{\text{grad}f}{|\text{grad}f|}$. Then

$$H = -\frac{1}{n}\text{div}\left(\frac{\text{grad}f}{|\text{grad}f|}\right)$$

where the gradient and the divergence are defined by $\bar{g}$.

(3) Recall that the cartesian coordinates (x, y, z) and the spherical coordinates (r, ϕ, θ) on $\mathbb{R}^3$ are related by

$$x = r\sin\phi\cos\theta, \quad y = r\sin\phi\sin\theta, \quad z = r\cos\phi.$$

Consider a Riemannian metric on $\mathbb{R}^3$ of the form

$$g = u(r)^2 dr^2 + r^2(d\phi^2 + \sin^2\phi\, d\theta^2)$$

where $u = u(r)$ is a positive, smooth function on $\mathbb{R}^3$ which depends only on r. Note that $u = 1$ corresponds to the Euclidean metric. Given any $\rho > 0$, let S_ρ be the sphere defined by $r = \rho$.

(a) Find the scalar curvature of $(\mathbb{R}^3, g)$.

(b) Find the second fundmantal form and the mean curvature of S_ρ with respect to the inner unit normal.

(c) Find the Gaussian curvature of S_ρ.