Mathematics G4402. Modern Geometry **Fall 2009**

Assignment 7

Due November 9, 2009

(1) Let H be a closed Lie subgroup of a Lie group G, and let $G/H = \{aH \mid a \in G\}$ be the set of *left* cosets of H in G. (In other words, let H act on G by *right* multiplication and let G/H be the quotient.) G acts on G/H on the *left* by $G \times G/H \to G/H$, $(a, bH) \mapsto abH$. Let g be a *right* invariant Riemannian metric on G. By the theorems stated in class, there is a unique Riemannian metric $\hat{g}$ on G/H such that $\pi : (G, g) \to (G/H, \hat{g})$ is a Riemannian submersion. Prove that if g is left invariant then G acts isometrically on $(G/H, \hat{g})$.

(2) Recall that for any $A \in SO(n) = \{A \in M_n(\mathbb{R}) \mid A^t A = I_n, \det A = 1\}$ we have $T_A SO(n) = \{B \in M_n(\mathbb{R}) \mid B^t A + A^t B = 0\}$. Define a Riemannian metric g_n on $SO(n)$ by $g_n(A)(B, C) = \mathrm{Tr}(B^t C)$ for any $A \in SO(n)$ and $B, C \in T_A SO(n)$.

(a) Prove that g_n is a bi-invariant metric on $SO(n)$.

(b) We have seen in class that there is a diffeomorphism
$$f : S^n \to SO(n+1)/SO(n).$$
Let $\hat{g}$ be the unique Riemannian metric on $SO(n+1)/SO(n)$ such that
$$\pi : (SO(n+1), g_{n+1}) \to (SO(n+1)/SO(n), \hat{g})$$
is a Riemannian submersion. Prove that $f^*\hat{g} = \lambda g_{\mathrm{can}}$ for some $\lambda > 0$, and find λ. (Hint: (i) $SO(n+1)$ acts isometrically on (S^n, g_{can}) and on $(SO(n+1)/SO(n), \hat{g})$; (ii) what is the horizontal space $H_{I_{n+1}} \subset T_{I_{n+1}} SO(n+1)$?)

(3) Let $F : (M, g) \to (N, h)$ be an isometric immersion. For any $p \in M$, let π_p be the orthogonal projection from $T_{F(p)}N$ to the image of $dF_p : T_p M \to T_{F(p)}N$. Let X, Y be C^∞ vector fields on M which are F-related to C^∞ vector fields $\tilde{X}, \tilde{Y}$ on N, respectively. Let ∇ and $\tilde{\nabla}$ be the Levi-Civita connections on (M, g) and on (N, h), respectively. Prove that for any $p \in M$,
$$dF_p\big((\nabla_X Y)(p)\big) = \pi_p\big((\tilde{\nabla}_{\tilde{X}} \tilde{Y})(F(p))\big)$$

(4) Let $\mathcal{H} = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_2 > 0\}$ be the upper half plane, and define a Riemannian metric on $\mathcal{H}$ by
$$g = \frac{dy_1^2 + dy_2^2}{y_2^2}.$$

(a) Compute the Christoffel symbols Γ_{ij}^k, $i, j, k \in \{1, 2\}$, for the Levi-Civita connection ∇ on $(\mathcal{H}, g)$.

(b) Define $\gamma : \mathbb{R} \to \mathcal{H}$ by $\gamma(t) = (t, 1)$. Then γ is a smooth curve in $\mathcal{H}$. Let
$$V(t) = a(t)\frac{\partial}{\partial y_1} + b(t)\frac{\partial}{\partial y_2}$$
be the unique parallel (w.r.t. ∇) vector field along γ such that $V(0) = \frac{\partial}{\partial y_2}$. Find $a(t), b(t)$ for $t \in \mathbb{R}$.