Mathematics G4402. Modern Geometry **Fall 2009**

Assignment 6

Due on Monday, October 26, 2009

[dC]= do Carmo, *Riemannian Geometry*
[GHL]= Gallot, Hulin, Lafontaine, *Riemannian Geometry*

(1) Let $\mathcal{H}^n = \{(y_1, \dots, y_n) \in \mathbb{R}^n \mid y_n > 0\}$ be the n-dimensional upper half space. Define a Riemannian metric h on $\mathcal{H}_n$ by

$$h = \frac{dy_1^2 + \cdots + dy_n^2}{y_n^2}.$$

Let D_n and $(f^{-1})^*g$ be defined as in Problem 1(b) of Assignment 5. Prove that $(\mathcal{H}^n, h)$ is isometric to $(D_n, (f^{-1})^*g)$. (Hint: see [GHL] page 57.)

(2) Let G_1 and G_2 be Lie groups, and let $e_1 \in G_1$ and $e_2 \in G_2$ be the identity elements. Suppose that $f : G_1 \to G_2$ is a group homomorphism and a smooth map. Prove that $df_{e_1} : T_{e_1}G \to T_{e_2}H$ is a Lie algebra homomorphism.

(3) Let

$$S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1},$$

and let g_{can} denote the Riemannian metric on S^n induced by the Euclidean metric on $\mathbb{R}^{n+1}$. Let $\hat{g}_{\text{can}}$ be the Riemannian metric on $P^n(\mathbb{C})$ such that $\pi : (S^{2n+1}, g_{\text{can}}) \to (P^n(\mathbb{C}), \hat{g}_{\text{can}})$ is a Riemannian submersion.

(a) Prove that $P^1(\mathbb{C})$ is diffeomorphic to S^2.

(b) Prove that $(P^1(\mathbb{C}), \hat{g}_{\text{can}})$ is isometric to $(S^2, \frac{1}{4}g_{\text{can}})$.

(4) Let G be a compact connected Lie group ($\dim G = n$).

(a) Let ω be a left invariant C^∞ n-form on G, that is, $L_x^*\omega = \omega$ for all $x \in G$. Prove that ω is right invariant. (Hint: see [dC] page 47.)

(b) Show that there exists a left-invariant C^∞ n-form on G.

(c) Let $\langle\ ,\ \rangle$ be a left invariant metric on G, and let ω be a left invariant C^∞ n-form on G such that $\int_G \omega > 0$. Define a new Riemannian metric $\langle\langle\ ,\ \rangle\rangle$ on G by

$$\langle\langle u, v\rangle\rangle_y = \int_G \langle (dR_x)_y(u), (dR_x)_y(v)\rangle_{yx}\, \omega, \qquad x, y \in G, \quad u, v \in T_yG.$$

Prove that the new Riemannian metric $\langle\langle\ ,\ \rangle\rangle$ is bi-invariant.