

Mathematics G4402. Modern Geometry
Assignment 3

Fall 2009

Due on Monday, October 5, 2009

An open interval in \mathbb{R} is of the form (a, b) , where $-\infty \leq a < b \leq +\infty$.

- (1) Let X be a smooth vector field on a smooth manifold M , and let $\gamma : I \rightarrow M$ be a nonconstant integral curve of X , where I is an open interval in \mathbb{R} . Prove the following statements.
 - (a) γ is an immersion.
 - (b) If γ is not injective, then there exists a smooth embedding $i : S^1 \rightarrow M$ such that $i(S^1) = \gamma(I)$.
- (2) Let X be the vector field on \mathbb{R} defined by $X(x) = x^2 \frac{\partial}{\partial x}$. Given $x \in \mathbb{R}$, let $\phi_x : I_x \rightarrow \mathbb{R}$ be the unique integral curve of X such that $\phi_x(0) = x$, where I_x is an open interval containing 0, and ϕ_x cannot be extended to a larger open interval containing I_x . Find ϕ_x and I_x for all $x \in \mathbb{R}$.
- (3) Let X, Y, Z be vector fields on \mathbb{R}^3 defined in (4) of Assignment 2. Compute the flow of the vector field $aX + bY + cZ$ where $a, b, c \in \mathbb{R}$.
- (4) Let M be a smooth submanifold of a smooth manifold N , and let X, Y be smooth vector fields on M . Let $p \in M$ and let U be an open neighborhood of p in N .
 - (a) Suppose that $\tilde{X}, \tilde{Y} \in C^\infty(U, TU)$ are smooth vector fields on U such that for all $q \in U \cap M$
$$\tilde{X}(q) = X(q) \in T_q M, \quad \tilde{Y}(q) = Y(q) \in T_q M.$$
Show that $[\tilde{X}, \tilde{Y}](q) \in T_q M$ for all $q \in U \cap M$.
 - (b) Let f be a smooth function on M , and let \tilde{f} be a smooth function on U such that $\tilde{f}(q) = f(q)$ for all $q \in U \cap M$. Let $g = [X, Y]f \in C^\infty(M)$ and let $\tilde{g} = [\tilde{X}, \tilde{Y}]\tilde{f} \in C^\infty(U)$. Show that $\tilde{g}(q) = g(q)$ for all $q \in U \cap M$.