Mathematics G4402. Modern Geometry **Fall 2009**

Assignment 2

Due on Monday, September 28, 2009

(1) Let $p(x_1, \dots, x_k) \in \mathbb{R}[x_1, \dots, x_k]$ be a homogeneous polynomial of degree m, i.e.,

$$p(tx_1, \dots, tx_k) = t^m p(x_1, \dots, x_k).$$

We assume that $m \geq 2$.

(a) Prove that if $a \neq 0$ then

$$X_a = \{x \in \mathbb{R}^k \mid p(x) = a\}$$

is a $k-1$ dimensional submanifold of $\mathbb{R}^k$. [Hint: Use Euler's identity for homogeneous polynomials

$$\sum_{i=1}^{k} x_i \frac{\partial p}{\partial x_i} = m \cdot p$$

to prove that 0 is the only critical value of p.]

(b) Prove that X_a is diffeomorphic to X_1 if $a > 0$, and X_a is diffeomorphic to X_{-1} if $a < 0$.

(2) Let $M_n(\mathbb{R})$ be the space of $n \times n$ matrices (with real entries). We assume that $n \geq 2$. Let $f : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ be defined by $f(A) = \det(A)$.

(a) Recall that the classical adjoint of $A \in M_n(\mathbb{R})$ is defined by

$$(\mathrm{adj}A)_{ij} = (-1)^{i+j} \det A(j \mid i)$$

where $A(j \mid i) \in M_n(\mathbb{R})$ is obtained by removing the j-th row and i-th column from A. Show that the differential of f at A is given by

$$df_A : M_n(\mathbb{R}) \to \mathbb{R}, \quad df_A(B) = \mathrm{Tr}\left((\mathrm{adj}A)B\right).$$

(b) Show that 0 is the only critical value of $f : M_n(\mathbb{R}) \to \mathbb{R}$. [Hint: $A(\mathrm{adj}A) = (\det A)I_n$, where $I_n \in M_n(\mathbb{R})$ is the identity matrix.]

(c) Let $SL(n) = \{A \in M(n) \mid \det A = 1\} = f^{-1}(1)$. By (b), $SL(n)$ is an (n^2-1)-dimensional submanifold of $M_n(\mathbb{R})$. Describe $T_{I_n}SL(n)$ (the tangent space to $SL(n)$ at the identity matrix I_n) explicitly as a linear subspace of $M_n(\mathbb{R})$.

(d) Describe $TSL(n)$ (the tangent bundle of $SL(n)$) explicitly as a subset of $M_n(\mathbb{R}) \times M_n(\mathbb{R})$.

(3) (universal line bundle) Recall that

$$P_n(\mathbb{R}) = \{\ell \subset \mathbb{R}^{n+1} \mid \ell \text{ is a 1 dimensional linear subspace of } \mathbb{R}^{n+1}\}.$$

Define

$$E = \{(\ell, v) \in P_n(\mathbb{R}) \times \mathbb{R}^{n+1} \mid v \in \ell\} \subset P_n(\mathbb{R}) \times \mathbb{R}^{n+1}.$$

Let $p_1 : P_n(\mathbb{R}) \times \mathbb{R}^{n+1} \to P_n(\mathbb{R})$ be the projection to the first factor, and let $\pi : E \to P_n(\mathbb{R})$ be the restriction of p_1 to E. Prove that $\pi : E \to P_n(\mathbb{R})$ is a C^∞ vector bundle of rank 1 over $P_n(\mathbb{R})$.

(4) Let X, Y, Z be the vector fields defined on $\mathbb{R}^3$ by

$$X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad Y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad Z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

So that the map $(a, b, c) \mapsto aX + bY + cZ$ is an isomorphism from $\mathbb{R}^3$ onto a subspace of the space of smooth vector fields on $\mathbb{R}^3$, and that the bracket of vector fields on $\mathbb{R}^3$ corresponds to the cross product on $\mathbb{R}^3$.