Mathematics G4402. Modern Geometry **Fall 2009**

Assignment 1

Due on Monday, September 21, 2009

In this assignment:

- smooth structure = C^∞ differentiable structure
- smooth map = C^∞ map

(1) (Stereographic projection) Let

$$S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$$

be equipped with the subset topology. Let $N = (0, \dots, 0, 1)$ and $S = (0, \dots, 0, -1)$ be the north and south poles of S^n, respectively. Define $\pi_1 : S^n - \{N\} \to \mathbb{R}^n$ (resp. $\pi_2 : S^n - \{S\} \to \mathbb{R}^n$) such that $(\pi_1(p), 0)$ (resp. $(\pi_2(p), 0)$) is the point at which the line through N (resp. S) and p intersects the hyperplane $\{x_{n+1} = 0\}$. (See do Carmo page 19 Example 4.6.)

(a) Prove that $\Phi = \{(S^n - \{N\}, \pi_1), (S^n - \{S\}, \pi_2)\}$ is a C^∞ atlas on S^n.

(b) Prove that the inclusion map $i : S^n \to \mathbb{R}^{n+1}$ is a smooth embedding, where S^n is equipped with the smooth structure defined by the C^∞ atlas Φ. Therefore the smooth structure defined by Φ coincides with the smooth structure on S^n as a submanifold of $\mathbb{R}^{n+1}$.

(2) (Embedding of $P^2(\mathbb{R})$ in $\mathbb{R}^4$) Let $F : \mathbb{R}^3 \to \mathbb{R}^4$ be given by

$$F(x, y, z) = (x^2 - y^2, xy, zx, yz).$$

Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. Observe that $f = F|_{S^2}$ satisfies $f(x, y, z) = f(-x, -y, -z)$, so that it descends to a map $\tilde{f} : P^2(\mathbb{R}) = S^2/\{\pm 1\} \to \mathbb{R}^4$. Prove that $\tilde{f}$ is a smooth embedding. [Hint: A bijective continuous map from a compact topological space to a Hausdorff topological space is a homeomorphism.]

(3) Let (x, y, z) be coordinates on $\mathbb{R}^3$. Let Y_r be the set of points in $\mathbb{R}^3$ at distance $r > 0$ from the circle

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, z = 0\}.$$

(a) Let $A = \{r \in (0, \infty) \mid Y_r \text{ is a smooth submanifold of } \mathbb{R}^3\}$. Find A.

(b) Let S^1 be equipped with the smooth structure in (1), and let $S^1 \times S^1$ be the product manifold (see do Carmo page 31-32 Exercise 1). Prove that Y_r is C^∞ diffeomorphic to $S^1 \times S^1$ for any $r \in A$.

(4) Prove that the tangent bundle TM of a smooth manifold M is an orientable manifold (even though M may not be).