Algebraic Bethe Ansatz and the XXX Spin Chain

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Motivation

The algebraic Bethe ansatz is great because of its universality: it is only based on the RLL relation and without any changes it can be applied to a wide class of models. In this way, different physical models are different representations of the same algebra. Bethe [1931] is where we get the (coordinate) Bethe ansatz; the method is for constructing the eigenfunctions of the quantum Hamiltonian of a Heisenberg spin chain. In the late 70s and early 80s, Faddeev et al [1979a], Faddeev et al [1979b], and Sklyanin [1982] developed the Quantum Inverse Scattering method. This framework showed that many quantum models solved by the Bethe ansatz that were completely different *physically* were the same *algebraically*. Thus, many important properties of physical systems can be established at the level of algebra, without using concrete representations. Explained briefly, the algebraic Bethe ansatz is a method of working with a special operator algebra describing a rather wide class of quantum systems.

0 Definitions

Lax matrix:

$$L_{an}(u) = \begin{pmatrix} u + iS_n^z & iS_n^- \\ iS_n^+ & u - iS_n^z \end{pmatrix}$$

Monodromy matrix:

$$M_a(u) = L_{a1}(u) \dots L_{aL}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

Transfer matrix:

$$T(u) = \mathrm{tr}_a M_a(u) = A(u) + D(u)$$

1 The algebra

The algebra between A, B, C, D is a consequence of RLL. It's given by the RMM relation, which we proved last week. To extract the algebra, we look at the monodromy matrix: the central object of the algebraic Bethe ansatz.

$$\widetilde{M}_a(u) = M_a(u) \otimes \mathbb{I}_b = \begin{pmatrix} A(u) & 0 & B(u) & 0 \\ 0 & A(u) & 0 & B(u) \\ C(u) & 0 & D(u) & 0 \\ 0 & C(u) & 0 & D(u) \end{pmatrix}_{a,b}$$

and

$$\widetilde{M}_b(v) = \mathbb{I}_a \otimes M_b(v) = \begin{pmatrix} A(v) & B(v) & 0 & 0 \\ C(v) & D(v) & 0 & 0 \\ 0 & 0 & A(v) & B(v) \\ 0 & 0 & C(v) & D(v) \end{pmatrix}_{a,b} .$$

These operators commute. We plug into the RMM relation and get

$$
\begin{aligned}
B(u)B(v) &= B(v)B(u), \\
A(u)B(v) &= f(v-u)B(v)A(u) + g(u-v)B(u)A(v), \\
D(u)B(v) &= f(u-v)B(v)D(u) + g(v-u)B(u)D(v),
\end{aligned}
$$

where $f(u) = \frac{u+i}{u}$ and $g(u) = \frac{i}{u}$. h, which comes up later, is defined as $h(u) = \frac{u-i}{u}$.

2 From transfer matrix to Hamiltonian

Two properties to exploit:

1. The Hamiltonian of the Heisenberg spin chain H_{XXX} is hidden inside the transfer matrix $T(u)$. Precisely,

$$H_{XXX} \sim \frac{d}{du} \log T(u)\Big|_{u=i/2}.$$

2. The eigenstate of the Hamiltonian H_{XXX} (and $T(u)$) can be constructed by

$$B(u_1)\dots B(u_N)|\uparrow\dots\uparrow\rangle$$

   where B is one of the components in the monodromy matrix.

2.1 Permutation operators

Consider the following operator acting on $\mathbb{C}^2 \otimes \mathbb{C}^2$:

$$P_{ab} = \frac{1}{2}\big(I_a \otimes I_b + \sum_\alpha \sigma_a^\alpha \otimes \sigma_b^\alpha\big), \qquad \alpha = x, y, z.$$

Alternatively,

$$P_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{ab}.$$

For any $|x\rangle_a, |y\rangle_b$,

$$P_{ab}(|x\rangle_a \otimes |y\rangle_b) = |y\rangle_a \otimes |x\rangle_b.$$

Both H_{XXX} and the Lax operator can be written in terms of these operators. Recall

$$H_{XXX} = \sum_{n=1}^{L} \vec{S}_n \cdot \vec{S}_{n+1} = \frac{1}{4}\sum_{n=1}^{L} \vec{\sigma}_n \cdot \vec{\sigma}_{n+1}.$$

Proceeding from the definition of P_{ab},

$$\vec{\sigma}_n \cdot \vec{\sigma}_{n+1} = 2P_{n,n+1} - I_{n,n+1} \implies H_{XXX} = \frac{1}{2}\sum_{n=1}^{L} P_{n,n+1} - \frac{L}{4}.$$

In order to write down the Hamiltonian, set $V_n = \mathbb{C}^2$. Thus, the Lax operator in terms of P_{ab} is given by

$$L_{an}(u) = (u - \frac{i}{2})I_{a,n} + iP_{a,n}.$$

This implies pretty useful observations:

- $L_{an}(\frac{i}{2}) = iP_{a,n}$.
- $\frac{d}{du}L_{an}(u) = I_{a,n}$

2.2 The shift operator

The shift operator U is given by

$$
\begin{aligned}
U &= i^{-L}T(\frac{i}{2}) = i^{-L}\mathrm{tr}_a M_a(\frac{i}{2}) \\
&= \mathrm{tr}_a P_{a,1}\dots P_{a,L} = (\mathrm{tr}_a P_{a,L})P_{L,L-1}\dots P_{1,2} \\
&= P_{L,L-1}\dots P_{1,2}.
\end{aligned}
$$

This is made easier since $\mathrm{tr}_a P_{a,n} = I_n$.

2.3 The Hamiltonian

Let's expand $T(u)$ around $u = \frac{i}{2}$. First,

$$
\frac{d}{du}M_a(u)\Big|_{u=\frac{i}{2}} = i^{L-1}\sum_{n=1}^{L} P_{a,1}\dots \hat{P_{a,n}}\dots P_{a,L},
$$

where $\widehat{P}_{a,n}$ means the permuation operator is omitted. To obtain the derivative with respect to the transfer matrix, take the trace in the auxiliary space:

$$
\begin{aligned}
\frac{d}{du}T(u)\Big|_{u=\frac{i}{2}} &= \frac{d}{du}(\mathrm{tr}_a(L_{a1}(u)\dots L_{aL}(u)))\Big|_{u=\frac{i}{2}} \\
&= \mathrm{tr}_a\left(\frac{d}{du}L_{a1}(u)\dots L_{aL}(u)\right)\Big|_{u=\frac{i}{2}} \\
&= \mathrm{tr}_a\left(\sum_{n=1}^{L} L_{a,1}\dots \hat{L_{a,n}}\dots L_{a,L}\right)\Big|_{u=\frac{i}{2}} \\
&= \sum_{k=1}^{L} i^{L-1}\mathrm{tr}_a(P_{a,1}\dots \hat{P_{a,n}}\dots P_{a,L}) \\
&= i^{L-1}\sum_{k=1}^{L} P_{L,L-1}\dots P_{n-1,n+1}\dots P_{1,2}.
\end{aligned}
$$

From here, consider $T(u)^{-1}$. Recall that $T(\frac{i}{2}) = i^L P_{L,L-1}\dots P_{2,1}$:

$$
\begin{aligned}
T^{-1}(\frac{i}{2}) &= (-i)^L P_{1,2}\dots P_{L,L-1} \\
\left(\frac{d}{du}T(u)\right)T(u)^{-1}\Big|_{u=\frac{i}{2}} &= \frac{1}{i}\sum_{n=1}^{L} P_{n,n+1}.
\end{aligned}
$$

The left hand side can be written as the logarithm derivative

$$
\left(\frac{d}{du}T(u)\right)T(u)^{-1} = \frac{d}{du}\log T(u).
$$

Plug this into the expression for the Hamiltonian:

$$
H_{XXX} = \frac{i}{2}\frac{d}{du}\log T(u)\Big|_{u=\frac{i}{2}} - \frac{L}{4}.
$$

3 Construction of eigenvectors

Let's turn to the eigenstates: they can be constructed by acting B-operators on the pseudovaccum $|\uparrow^L\rangle$.

3.1 Pseudovacuum

The pseudovacuum state $|\uparrow^L\rangle$ diagonalizes $A(u)$ and $D(u)$ and is anhilated by $C(u)$. Consider $M_a(u)|\uparrow^L\rangle$:

$$\begin{aligned} M_a(u)|\uparrow^L\rangle &= L_{a1}(u)\dots L_{aL}(u)|\uparrow^L\rangle \\ &= (L_{a1}(u)|\uparrow\rangle_1)\otimes\dots\otimes(L_{aL}(u)|\uparrow\rangle_L). \end{aligned}$$

On each site,

$$L_{an}(u)|\uparrow\rangle_n = \begin{pmatrix} u+iS_n^z & iS_n^- \\ iS_n^+ & u-iS_n^z \end{pmatrix}|\uparrow\rangle_n = \begin{pmatrix} u+\frac{i}{2} & iS_n^- \\ 0 & u-\frac{i}{2} \end{pmatrix}|\uparrow\rangle_n.$$

So at each site, we have an upper triangular matrix: the multiplication of upper triangular matrices is still an upper triangular matrix. Thus,

$$\begin{aligned} M_a(u)|\uparrow^L\rangle &= \begin{pmatrix} u+\frac{i}{2} & iS_1^- \\ 0 & u-\frac{i}{2} \end{pmatrix}\cdots\begin{pmatrix} u+\frac{i}{2} & iS_L^- \\ 0 & u-\frac{i}{2} \end{pmatrix}|\uparrow^L\rangle \\ &= \begin{pmatrix} (u+\frac{i}{2})^L & \star \\ 0 & (u-\frac{i}{2})^L \end{pmatrix}|\uparrow^L\rangle \end{aligned}$$

Compare this to

$$M_a(u)|\uparrow^L\rangle = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}|\uparrow^L\rangle.$$

This gives us

$$A(u)|\uparrow^L\rangle = a(u)|\uparrow^L\rangle, \qquad D(u)|\uparrow^L\rangle = d(u)|\uparrow^L\rangle, \qquad C(u)|\uparrow^L\rangle = 0,$$

where $a(u) = (u-\frac{i}{2})^L$ and $d(u) = (u+\frac{i}{2})^L$.

Clearly, $|\uparrow^L\rangle$ is an eigenvector of $A(u)$ and $D(u)$; $C(u)$ is eliminated as prophesied. Such a state is also called a highest weight state or reference state. The existence of such a state is non-trivial and is a necessary condition that the Bethe ansatz works. Some integrable models don't have this (XYZ, Toda, etc.); these models need to use other methods, like Sklyanin's separation of variables.

3.2 The N-magnon state

We conveniently left $B(u)$ out of the picture. $B(u)$ acting on $|\uparrow^L\rangle$ is pretty complicated. Each $B(u)$ on a given state flips down a spin. The flipped spin can be located at any site of the spin chain which need to be summed over with different weights. Now we'll show that

$$|\mathbf{u}_n\rangle = B(u_1)\dots B(u_N)|\uparrow^L\rangle$$

is an eigenstate of $T(u)$ if $\mathbf{u}_N$ satisfies certain conditions.

First let's see how $A(u)$ and $D(u)$ act on $|\mathbf{u}_N\rangle$. At $N=1$ (the 1-magnon sector), $A(u)|\mathbf{u}_1\rangle$ is pretty straightforward:

$$\begin{aligned} A(u)|\mathbf{u}_1\rangle &= A(u)B(u_1)|\uparrow^L\rangle \\ &= (f(u_1-u)B(u_1)A(u)+g(u-u_1)B(u)A(u_1))|\uparrow^L\rangle \\ &= f(u_1-u)a(u)|\uparrow^L\rangle + g(u-u_1)a(u_1)B(u)|\uparrow^L\rangle. \end{aligned}$$

. $D(u)|\mathbf{u}_1\rangle$ is almost exactly the same.

Generalizing to N from here yields

$$\begin{aligned} A(u)|\mathbf{u}_N\rangle &= A(u)B(u_1)\dots B(u_N)|\uparrow^L\rangle \\ &= a(u)\prod_{k=1}^{N} f(u_k-u)B(u_1)\dots B(u_N)|\uparrow^L\rangle + \sum_{k=1}^{N} M_k(u|\mathbf{u}_N)B(u_1)\dots\widehat{B}(u_k)\dots B(u_N)B(u)|\uparrow^L\rangle, \end{aligned}$$

where $\widehat{B}(u_k)$ means the operator is omitted. The first term takes the form of an eigenstate and is called the "wanted term". The second terms are called "unwanted terms". The coefficients $M_k(u|\mathbf{u}_N)$ can be determined using the algebra established:

$$M_1(u|\mathbf{u}_N) = g(u-u_1)a(u_1)\prod_{k=2}^{N} f(u_k-u_1).$$

Since all the B-operators commute, $M_k(u|\mathbf{u}_N)$ comes from a substitution of u_k in place of u_1:

$$M_j(u|\mathbf{u}_n) = g(u-u_j)a(u_j)\prod_{k\neq j}^{N} f(u_k-u_j).$$

For D,

$$D(u)B(u_1)\dots B(u_N)|\Omega\rangle = d(u)\prod_{k=1}^{N} f(u-u_k)B(u_1)\dots B(u_N)|\uparrow^L\rangle$$
$$+\sum_{j=1}^{N} N_j(u|\mathbf{u}_N)B(u_1)\dots\widehat{B}(u_k)\dots B(u_N)B(u)|\uparrow^L\rangle$$

where $N_j(u|\mathbf{u}_N) = g(u_j-u)d(u_j)\prod_{k\neq j}^{N} f(u_j-u_k)$.

$g(u-u_j) = -g(u_j-u)$, so let's take the sum of A and D: we get that

$$(A(u)+D(u))|\mathbf{u}_N\rangle = \tau(u|\mathbf{u}_N)|\mathbf{u}_n\rangle$$

where $\tau(u|\mathbf{u}_N)$ is the eigenvalue of the transfer matrix:

$$\tau(u|\mathbf{u}_N) = a(u)\prod_{j=1}^{N}\frac{u-u_j-i}{u-u_j} + d(u)\prod_{j=1}^{N}\frac{u-u_j+i}{u-u_j}.$$

The unwated terms cancel out under the condition

$$a(u_j)\prod_{k\neq j}^{N} f(u_j-u_k) = d(u_j)\prod_{k\neq j}^{N} h(u_j-u_k), \qquad j=1,\dots,N.$$

More explicitly,

$$\left(\frac{u_j+\frac{i}{2}}{u_j-\frac{i}{2}}\right)^L \prod_{k\neq j}^{N}\frac{u_j-u_k-i}{u_j-u_k+i} = 1, \qquad j=1,\dots,N.$$

Doesn't this look familiar?