Transfer Matrix

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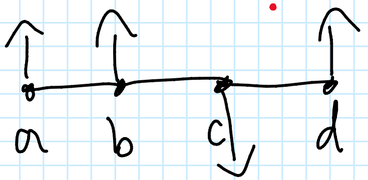
Def A stat mech system Π is a collection of states $\{\sigma\} = (\vec{\sigma}_x)_{x \in P}$, $\vec{\sigma}_x \in \mathbb{R}^d$, P = sites with a Hamiltonian H: States $\longrightarrow \mathbb{R}$ (energy)

Remark: The dimension of Π = dim of states, not the vectors!

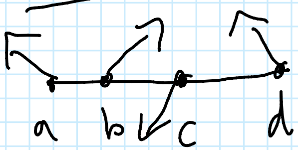
Ex (1D-Ising)



$\vec{\sigma}_\alpha \in \{\pm 1\}$

$\rightarrow$ line is 1D

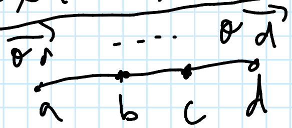
Ex (O(2)-model)



$\vec{\sigma}_\alpha \in S^1 = \{x_1^2 + x_2^2 = 1\}$ $\leftarrow$ vectors are 2D

$\longrightarrow$ line is 1D $\rightarrow$ dim $\Pi = 1$

Ex (O(n)-model)



$\vec{\sigma}_\alpha \in S^{n-1} = \{x_1^2 + \dots + x_n^2 = 1\}$

$\longrightarrow$ still a 1D system

Remark: O(1)-model = 1D-Ising as

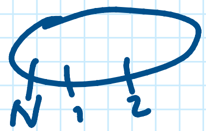
$S^0 = \{x_1^2 = 1\} \Rightarrow x_1 = \pm 1$

Def $V(\sigma_i, \sigma_j) = e^{J\sigma_i\sigma_j + \frac{1}{2}B(\sigma_i + \sigma_j)}$

Transfer matrix Sol to 1D-Ising

- periodic boundary conditions $\sigma_i = \sigma_{N+i}$

$$Z_N = \sum_{\{\sigma\}} e^{\sum_{i=1}^{N} J\sigma_i\sigma_{i+1} + \sum_{i=1}^{N} B\sigma_i}$$



Alg

$$= \sum_{\{\sigma\}} V(\sigma_1,\sigma_2)V(\sigma_2,\sigma_3)\dots V(\sigma_N,\sigma_1)$$

$$\overset{①}{=} \operatorname{Tr} V^N, \quad V = \begin{array}{c|cc} & 1 & -1 \\ \hline 1 & e^{J+B} & e^{-J} \\ -1 & e^{-J} & e^{J-B} \end{array}$$

Q: Why is ① true?

- Replace $\sigma_1 \leftrightarrow a$, $\sigma_2 \leftrightarrow b$, ... $\Rightarrow$

$$Z_N = \sum_{\alpha \in \{\pm 1\}} V(a,b)V(b,c)\dots V(\alpha_N, a)$$

Consider $V = \begin{pmatrix} V_{11} & V_{11'} \\ V_{1'1} & V_{1'1'} \end{pmatrix}$

Transfer Matrix 2

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$$V^2 = \begin{pmatrix} V_{11} & V_{11'} \\ V_{1'1} & V_{1'1'} \end{pmatrix}\begin{pmatrix} V_{11} & V_{11'} \\ V_{1'1} & V_{1'1'} \end{pmatrix}$$

$$= \begin{pmatrix} V_{11}V_{11} + V_{11'}V_{1'1} & V_{11}V_{11'} + V_{11'}V_{1'1'} \\ V_{1'1}V_{11} + V_{1'1'}V_{1'1} & V_{1'1}V_{11'} + V_{1'1'}V_{1'1'} \end{pmatrix}$$

$$\Rightarrow \operatorname{Tr}(V^2) = \sum_{b\in\{1,1'\}} V(1,b)V(b,1) + \sum_{b\in\{1,1'\}} V(1',b)V(b,1') = \sum_{a,b\in\{1,1'\}} V(a,b)V(b,a)$$

Lem $(V^N)_{ij} = \sum_{\alpha\in\{1,1'\}} V_{ib}V_{bc}V_{cd}\cdots V_{\alpha_N j}$

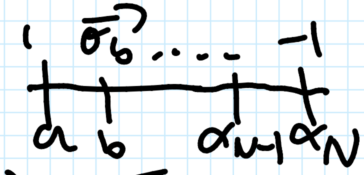
"N-th ele of alphabet"

Pf: Use induction.

Cor $\operatorname{Tr} V^N = \sum_{\alpha\in\{1,1'\}} V_{ab}V_{bc}\cdots V_{\alpha_N a}$

$\Rightarrow$ (1) $Z_N = \operatorname{Tr} V^N$

Remark: For non-periodic boundary conditions have to specify boundary of 1D-Ising, e.g.



$$\Rightarrow Z_N(1,-1) = \sum_{\{\sigma\}} V(1,b)V(b,c)\cdots V(\alpha_{N-1},-1)$$

Lem $= (V^{N-1})_{1,-1}$ entry "$=$" $(V^N)_{12}$ entry

$= (1\ \ 0)\, V^{N-1}\begin{pmatrix}0\\1\end{pmatrix} =$ (Let $PDP^{-1} = V$) $(\vec{v}_1, \vec{v}_2)$ e.v. of V

$(1\ \ 0)\, P \begin{pmatrix}\lambda_+^{N-1} & 0 \\ 0 & \lambda_-^{N-1}\end{pmatrix} P^{-1}\begin{pmatrix}0\\1\end{pmatrix} = \sin\phi\cos\phi\,(\lambda_+^{N-1} - \lambda_-^{N-1})$

eivalues of V

Periodic
↓

Series Expansion

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1D Ising, $B=0$: $Z_N(T) = \sum_{\{\theta\}} \prod_{i=1}^{N} e^{J\theta_i\theta_{i+1}}$

(a) $e^x = \cosh x + \sinh x$

$$\Rightarrow e^{J\theta_i\theta_j} = \cosh J\theta_i\theta_j + \sinh J\theta_i\theta_j$$

$\theta_i \in \{\pm 1\}$

$$= \underbrace{\cosh J}_{\text{even}} + \theta_i\theta_j \underbrace{\sinh J}_{\text{odd}}$$

$$= \cosh J\,(1 + \theta_i\theta_j \overset{v}{\overset{\parallel}{\tanh J}})$$

$$\Rightarrow Z_N(T) = \cosh^N J \sum_{\{\theta\}} \prod_{i=1}^{N} (1 + \theta_i\theta_{i+1} v)$$

$$= \cosh^N J \sum_{\{\theta\}} \sum_{a_i \in \{0,1\}} L_1^{a_1} L_2^{a_2} \cdots L_N^{a_N}\, v^{\sum a_i}, \qquad L_i = \theta_i\theta_{i+1}$$

switch Σ

$$= \cosh^N J \sum_{a_i \in \{0,1\}} \sum_{\{\theta\}} L_1^{a_1} L_2^{a_2} \cdots L_N^{a_N}\, v^{\sum a_i}$$

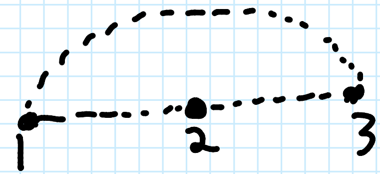
Def A nearest neighbor stat mech system is a graph $G = (V, E)$ w/ a collection of states $\{\theta\} = (\vec{\theta}_x)_{x \in V}$ w/ H: States $\rightarrow \mathbb{R}$ s.t. some function

$$H(\{\theta\}) = \sum_{(x,y) \in E} F(\vec{\theta}_x, \vec{\theta}_y)$$

– e.g. this rigorously defines nearest neighbor

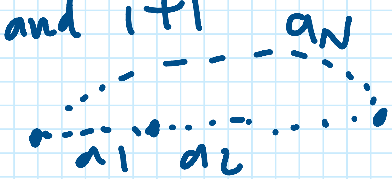
WARNING: Should draw edges of G as dashed lines

Ex: $N=3$ of 1D Periodic Ising



Key Point: If $a_i = 1$ fill in the line between vertex i and $i+1$

$L_1^{a_1} \cdots L_N^{a_N} \longleftrightarrow$



Series Expansion 2

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Claim 1: $\sum_{s=\{\theta\}} \theta_1^{c_1}\cdots\theta_n^{c_n} = \begin{cases} 2^N & \text{if all } c_i \text{ even} \\ 0 & \text{else} \end{cases}$

Pf: WLOG if $c_1=1$, then b/c $\theta_1\in\{\pm1\}$

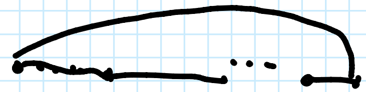
$$S = \sum_{\{\theta\}} 1^1\cdots\theta_n^{c_n} + \sum_{\{\theta\}} (-1)^1\cdots\theta_n^{c} = 0$$

$\Rightarrow$ all c_i even, but then as $\theta_i\in\{\pm1\}$

$$S = \sum_{\{\theta\}} 1 = 2^N$$

Claim2: All c_i even $\Longleftrightarrow$ closed polygons in lines corr to $L_1^{a_1}\cdots L_N^{a_N}$

Pf: Let i s.t. $a_i \neq 1$, $a_j=0$ for $j<i$. b/c $L_i=\theta_i\theta_{i+1}$, only L_i, L_{i+1} contribute a factor of θ_{i+1}. b/c $a_i\in\{0,1\} \Rightarrow a_{i+1}=1 \rightsquigarrow L_iL_{i+1}\cdots$

repeat $\Rightarrow$ all $a_i=1 \rightarrow$ 

Else, $\cancel{\exists}$ i s.t. $a_i\neq 1 \Rightarrow a_j=0\ \forall j$

$\Rightarrow$ empty polygon 

$$\Rightarrow Z_N(T) = \cosh^N J \sum_{a_i\in\{0,1\}} \sum_{\{\theta\}} L_1^{a_1}\cdots L_N^{a_N} v^{\sum a_i}$$

$$= \cosh^N J \sum_{\text{closed } P} \sum_{\{\theta\}} L_1^{a_1}\cdots L_N^{a_N} v^{\sum a_i}$$

$$\stackrel{C1+C2}{=} \cosh^N J\left(\sum_{\{\theta\}} \underbrace{L_1^1\cdots L_N^1}_{=\theta_1^2\cdots\theta_N^2} v^N + \sum_{\{\theta\}} 1\right)$$

$$\stackrel{C1}{=} \cosh^N J\left(2^N v^N + 2^N\right)$$ works

Remarks: Z_N "=" $\sum_{\text{closed polygon } P \text{ on } E} G(P)$

for any NN-stat mech system s.t. $F(\vec\theta_x,\vec\theta_y) = k_{xy}\vec\theta_x\vec\theta_y$, $\vec\theta_x\in\{\pm1\}\ \forall x\in V$, such as 2D-Ising

k_{xy} ← coupling constant

Duality in 2D Ising

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- If $k_{xy} = k \ \forall x,y$, then

$$Z_N(T) \text{``}=\text{''} \sum_{\text{closed } P \text{ on } E} (\# P \text{ w/ } k \text{ edges}) \, v^K$$

critical pt

Def Phase transition at $T_0 \Leftrightarrow Z_N(T_0) = \infty$

Goal: WTS 2D-Ising has phase transition.

Idea: 2D-Ising has duality as $N \to \infty$

Let $k_{xy} = k \ \forall x,y$, $k = J/T$

$$\frac{Z_N[k]}{(\sinh 2K)^{N/2}} = \frac{Z_N[\hat{k}]}{(\sinh 2\hat{k})^{N/2}} \quad (*)$$

high temp ← low temp

where $\tanh K = e^{-2\hat{K}}$ $(**)$

- high temp = converges as $N \to \infty$, $T \to \infty$
- low temp = __________, $T \to 0$

Let k_c = a critical point of $Z_N[k]$,

$\hat{k}_c$ = a critical point of $Z_N[\hat{k}]$

$(*) \Rightarrow \hat{k} = k_c$ is a critical point of $Z_N[\hat{k}]$ b/c $(\sinh 2k_c)^{N/2} < \infty$, physics says only one critical pt $\Rightarrow \hat{k}_c = k_c$ (1)

$(**) \Leftrightarrow \sinh 2\hat{k} \sinh 2k = 1$

$\overset{(1)}{\Rightarrow} \sinh 2k_c = 1$

Exer: $\sinh(\ln(1+\sqrt{2})) = 1$

$\Rightarrow 2k_c = \ln(1+\sqrt{2})$

$\Rightarrow T_c = \dfrac{2}{\ln(1+\sqrt{2})} J$