Pfaffians and the Dimer solution of the 2D−Ising Model

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OVERVIEW:

1. Motivation and Plan
2. Define a Pffafian
3. Prove detM = $[\text{pfM}]^2$
4. The Pfaffian and Dimer Configurations
5. The Dimer Solution

1. Motivation and Plan

Last talk Harrison defined the high-temperature expansion of the partition function of the 2D Ising Model. In this talk we will be introducing a combinatorial solution to this called the *Dimer Method.* I will start by defining a new mathematical construct known as the *pfaffian* of a matrix, and illustrate the concept of *dimer configurations*. Then I will show the relationship that exists between calculating the pfaffian of a matrix and counting the number of dimers on a square Ising lattice. We will then see how the dimer covering of the square Ising lattice has a one-to-one correspondence with the graphs of the Fisher lattice. Finally, we look into how the number of dimers on the Fisher lattice simplifies the computation of the high-temperature expansion of the partition function (Z_n) of the 2D Ising Model.

2. What is the Pfaffian?

Pfaffian → mathematical construct associated with antisymmetric matrices. In other words, it is really just a number we assign to a matrix that tells us info about it/its entries, in the same way a determinant would.

The pfaffian of a 2Nx2N antisymmetric matrix A is defined as:

$$\mathrm{Pf}A = \sum_P{}' \delta_P \, a_{p_1,p_2} \, a_{p_3,p_4} \cdots a_{p_{2N-1},p_{2N}},$$

Where A is the antisymmetric matrix defined below:

$$A = \begin{pmatrix} 0, & a_{1,2}, & \cdots, & a_{1,2N} \\ -a_{1,2}, & 0, & \cdots, & a_{2,2N} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ -a_{1,2N}, & -a_{2,2N}, & \cdots, & 0 \end{pmatrix}$$

Breaking down the equation we have:

1) **δ_P** = +/- 1 depending on if whether the individual permutation on P is obtained by an odd/even transpositions
2) **$P_1 \ldots P_{2N}$** = is a permutation on the set of numbers 1…2N
3) $\sum_P{}'$ = the sum over all the permutations that satisfy the conditions:
    a) $P_1<P_2$, $P_3<P_4$, … $P_{2N-1}<P_{2N}$
    b) $P_1<P_3<P_5 \ldots <P_{2N-1}$

Ex: Find the Pfaffian of the 4x4 antisymmetric matrix

Breaking down the problem we can ask: What are all the permutations satisfying the rules in red when N = 2?

Remember a way we learned to present permutations in group theory are the original ordering of values on top and their permutation below.

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ P_1 & P_2 & P_3 & P_4 \end{pmatrix}$$

Conditions:

$P_1<P_2$, $P_3<P_4$

$P_1<P_3$

Using the rules we can conclude that:

1. $P_1 = 1$ because nothing can be less than p1
2. $P_1< P_3 < P_4$ and $P_1< P_2$
3. Once P_2 is chosen, rest of the ordering is determined

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

$$\delta = 1 \qquad \delta = (-1)^1 = -1 \qquad \delta = (-1)^2 = 1$$

This is because of a crucial fact throughout this talk which seems trivial:

****<u>There exists a unique way totally order N numbers</u>****

Answer: **$Pf(A) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$**

3. Prove detM = [pfM]^2

Now this is not the most efficient way to find the pfaffian of a matrix. Let's prove that there exists a much simpler formula for the pfaffian.

THEOREM: If M is an even-dimensional complex non-singular 2N × 2N antisymmetric matrix, then there exists a non-singular 2N × 2N matrix P such that: $M = P^T J P$

Where J is the block diagonal matrix:

$$J = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & & 0 \\ & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \\ & & \ddots & \\ 0 & & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix}$$

We can use our formula for the Pfaffian to define the Pfaffian of J as:

$$Pf(J) = \sum_{P}{}' (-1)^{sign(P)} J_{P1,P2} J_{P3,P4} \dots J_{P2N-1,P2N}$$

Looking at the entries of our defined matrix, notice that:

$$J_{p_i,p_j} = \begin{cases} 0 & \text{if } P_i,P_j \neq (a, a+1) \text{ or if } a \text{ is even} \\ 1 & \text{if } P_i,P_j = (a, a+1) \text{ and } a \text{ is odd} \end{cases}$$

Therefore we can conclude that:

$Pf(J) = J_{P1,P2}J_{P3,P4}\ldots J_{P2N-1,P2N}$

$Pf(J) = \mathbf{1}$

Also note that:

det(J) = det(block diagonal)

$\det(J) = \Pi\ det(blocks)$

$\det(J) = \det \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{n}$

$\det(J) = 1^n$

$\det(J) = 1$

<u>THEOREM:</u> Given an arbitrary 2N × 2N complex matrix B and complex antisymmetric 2N × 2N matrix M, the following identity is satisfied: $pf(BMB^T)= pf(M)det(B)$

Now we have what we need to prove that there exists a relationship between the determinant and the pfaffian of an antisymmetric matrix

<u>PROVE:</u> $det(M) = [pf(M)]^2$

1. Assume that M is a non-singular complex 2n × 2n antisymmetric matrix
2. Start with the previous theorem we assumed to be true:

   $pf(BMB^T) = pf(M)det(B)$
3. Square both sides:

   $[pf(BMB^T)]^2 = pf(M)^2det(B)^2$
4. And from linear algebra we know that:

   $det(BMB^T) = det(M)det(B)^2$

5. We assumed M to be non-singular such that det(M) != 0. And if we assume B is a non-singular matrix, then we may the divide equations from step 3 and step 4:

$$\frac{[pf(BMB^T)]^2}{det(BMB^T)} = \frac{pf(M)^2 det(B)^2}{det(M)det(B)^2} =$$

$$\frac{[pf(BMB^T)]^2}{det(BMB^T)} = \frac{[pf(M)]^2}{det(M)}$$

6. Since this equation is true for any non-singular matrix B, we can choose a matrix B that allows us to trivially evaluate the left-hand-side of this equation, in this case: $B = (P^{-1})^T$

7. Using $B = (P^{-1})^T$ and the first theorem we defined: $M=P^TJP$ allows us to replace BMB^T with J

$$\frac{[pf(J)]^2}{det(J)} = \frac{[pf(M)^2]}{det(M)}$$

8. We already proved that the pf(J)= det(J) =1 →

$$\frac{[pf(M)^2]}{det(M)} = 1$$

9. We have proved that:

$$\mathbf{pf(M)^2 = det(M)}$$

4. The Pfaffian and Dimer Configurations

Now let's move to the next main portion of this talk which is **dimer configurations** and then define the relationship this with the Pfaffian.

DEF: A **dimer** is an object that can cover the links between nearest-neighbor sites, with the condition that a given site cannot be occupied by more than one dimer. It is like covering a graph with pairs of neighboring vertices with no overlap → also referred to in math as a ***perfect matching***

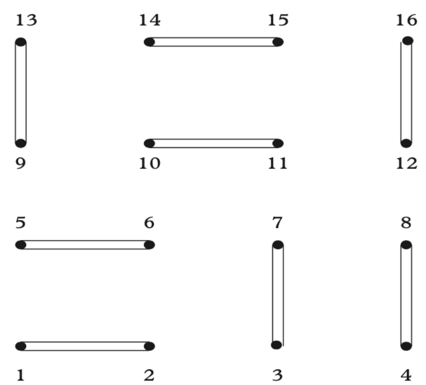
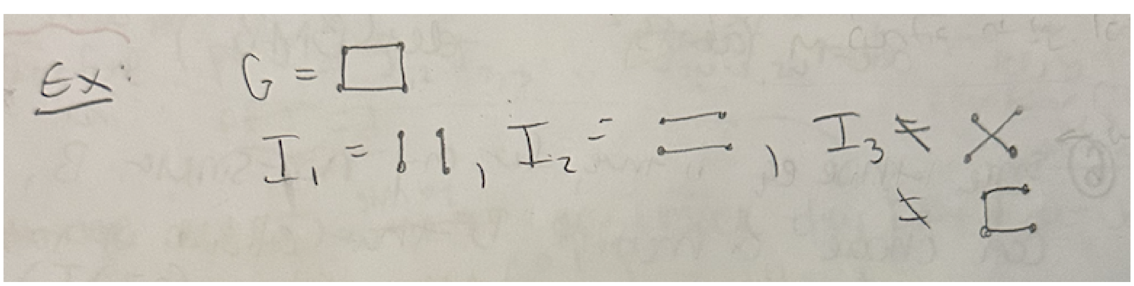


Fig. 5.9 *Dimer configuration of a* 4×4 *square lattice.*

DEF: Given graph G = (V,E), a dimer on G is a subset $\mathrm{I} \subseteq$ E such that each vertex v $\subseteq$ V is connected to one edge in I. The key example we will be working in this section with is the dimer configuration of the 2 x 2 square lattice

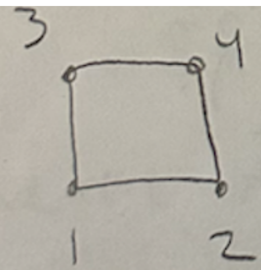


DEF: Given graph G, adjacent matrix A_G can be defined as:

$$(A_G)_{ij} = \begin{cases} 1 & \text{if } i \text{ — } j \\ 0 & \text{else} \end{cases}$$

Using our same example:

Ex:



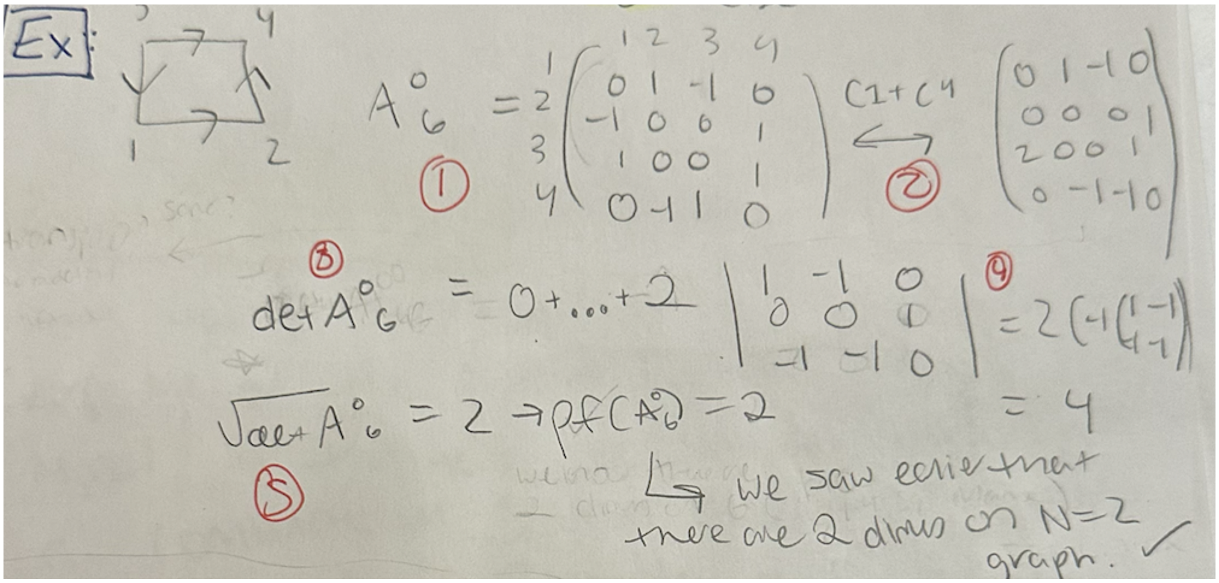
$$A_G = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

THEOREM: Kasteleyn's Theorem: For any planar (no overlapping edges) graph G, there exists an orientation on the edges of G such that: **pf(A^O_G) = number of dimers on G**

DEF: GIven oriented graph G (edges have orientation):

$$(A^O_G)_{ij} = \begin{cases} 1 & \text{if } i \rightarrow j \\ -1 & \text{if } i \leftarrow j \\ 0 & \text{else} \end{cases}$$

Now let’s calculate the pfaffian and see that Kasteleyn’s Theorem holds:



$\boxed{Ex}$: 3 4 1 2

$$A^o_6 = \begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix} \end{array} \xleftrightarrow{C1+C4} \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$$

1 2

3 $$\det A^o_6 = 0 + \dots + 2 \begin{vmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{vmatrix} = 2\left(-1\begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix}\right) = 4$$ 4

$$\sqrt{\det A^o_6} = 2 \to pf(A^o_6) = 2$$

5 ↳ We saw earlier that there are 2 dimers on N=2 graph. ✓

We can generalize our equation for the entries in the adjacency matrix A

$$A_{p,p'} = \begin{cases} z_1, & \text{for the horizontal links that are nearest-neighbour} \\ (-1)^p z_2, & \text{for the vertical links that are nearest-neighbour} \\ 0, & \text{otherwise} \end{cases}$$

It is easy to see that there is a one-to-one correspondence between the dimer configurations and the terms present in the definition of the Pfaffian of A

5. The Dimer Solution

Finally we claim that there is a one to one correspondence between the closed graphs of the high-temperature expansion of the 2D Ising model on a square lattice and the dimer configurations on the **Fisher lattice**. Now we can see that the number of dimers on the Fisher lattice allow us to compute the high-temperature expansion of the partition function of the 2D Ising Model.

Calculating the pfaffian = number of dimers 2D square Ising Lattice

$$\leftrightarrow$$

number of dimers on the Fisher lattice = Z_n of the 2D Ising Model

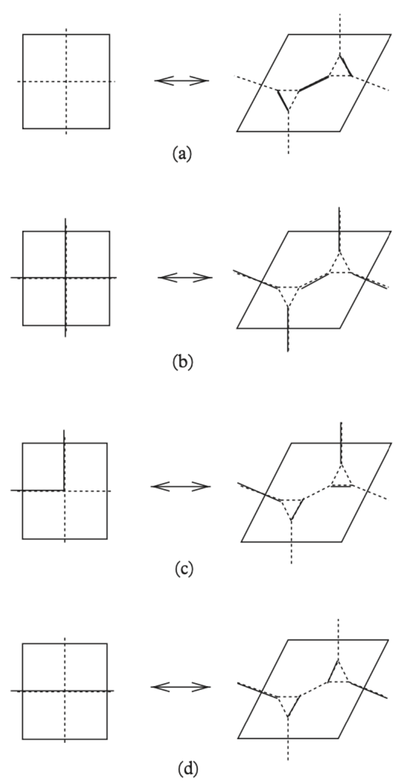


Fig. 5.14 *Correspondence between the lines of the Ising model on a square lattice and the dimers on the Fisher lattice.*

Let’s recall what Harrison defined last week:

$$(2\cosh K \cosh L)^{-N} Z_N = \sum_{r,s=0}^{\infty} n(r,s) v^r w^s,$$

Where n(r,s) is the number of closed polygons with r horizontal and v vertical links.

Assigning weight v to the dimers along the segments a1 and a3, weight w to the dimers placed on the segments a2 and a4, and weight 1 to all internal dimers of the cell, it is easy to see that the right-hand side of the above equation can also be interpreted as the number of dimer configurations on the Fisher lattice, and therefore this function can be expressed in terms of the Pfaffian of an anti-symmetric matrix A.