Outline:

- Physics Motivation
- Hyperbolic Trig Review
- The Ising Model Setup
- Solving the Ising Model:
    - The Recursive Method
    - Transfer Matrix Method

Physics Motivation

In Physics, Statistical Mechanics is the process where we start with a description of a physical system and end with macroscopic observables that we can measure.

If you think back to early high school or college physics, you typically write down the equations of motion or the Hamiltonian for a single particle or a system of a few particles. And by solving these equations you can understand the behavior of the system. But when we deal with larger systems like solids, liquids, and gasses, that contain a very large number of particles, figuring out all the individual equations of motion would be impossible. So that's why we need something like statistical mechanics to study the macroscopic properties of matter.

So before we dive into analyzing the Ising Model itself, I am going to review the hyperbolic trig functions, because their identities will show up often as we are dealing with lots of exponential functions.

Hyperbolic Trig Review

You have probably encountered these hyperbolic trig functions several times in the past, but if you are like me, you have most likely forgotten them.

So the trig functions that are going to be relevant in our work today are:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

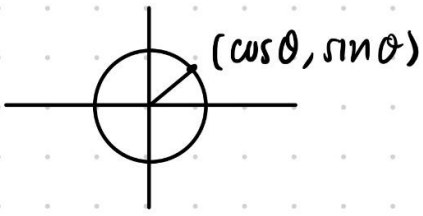
$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Why are they called hyperbolic trig functions?

Name Explanation

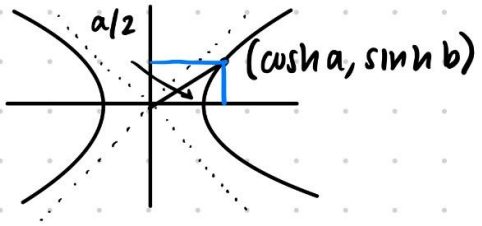
Hyperbolic trig functions are analogues to the normal trig functions. Like a circle can be parameterized using regular trig functions, a hyperbola can be parameterized with hyperbolic trig functions:

Normal Trig: Trace out unit circle:



Eq: $x^2+y^2=1$

Hyperbolic Trig: Trace out unit hyperbola



Eq: $x^2-y^2=1$

Like how normal trig functions trace out a unit circle, with the coordinates of that circle being defined by (cos θ, sin θ), hyperbolic trig functions trace out a unit hyperbola with coordinates defined by (cosh θ, sinh θ).

And you can already see and identity emerging in the equation for the unit hyperbola, which is $x^2 - y^2 = 1$, or since $x = \cosh \theta$ and $y = \sinh \theta$, this can also be written as $\cosh^2(x) - \sinh^2(x) = 1$

This makes sense geometrically, but I will also present the algebraic proof.

Proof of $\cosh^2(x) - \sinh^2(x) = 1$

Proof: $\cosh^2(x) - \sinh^2(x) = 1$

$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{1}{4}(e^x + e^{-x})^2 - \frac{1}{4}(e^x - e^{-x})^2 \qquad \text{Pull out } (\tfrac{1}{2})^2$$

$$= \frac{1}{4}(e^{2x} + 2e^x e^{-x} + e^{-2x}) - \frac{1}{4}(e^{2x} - 2e^x e^{-x} + e^{-2x}) \qquad \text{expand, cross terms cancel}$$

$$= \frac{1}{4}e^{2x} + \frac{1}{2} + \frac{1}{4}e^{-2x} - \frac{1}{4}e^{2x} + \frac{1}{2} - \frac{1}{4}e^{-2x}$$

$$= \frac{1}{2} + \frac{1}{2}$$

$$= 1$$

Even though this is certainly not the first time you have seen this proof, it is worth reviewing because it is an identity that we will be using a lot, as the Ising model deals with a lot of exponentials that are simplified using the hyperbolic trig functions.

Now, let's move on to talking about the Ising Model.

The Ising Model

The Ising Model is the simplest model for understanding phase transitions. It is so useful in physics because it is an exactly solvable model, and it can be applied universally to many different phase transitions, such as going in between solids, liquids, and gasses, magnets, and superconductors.

In one dimension, the Ising model describes a series of adjacent sites, where each site can be one of two states. Each site labeled with the index *i* and state σ. (picture)

These states can represent basically anything. Last time we talked about magnetism, so if we think of each site as a magnet moment, $\sigma = +1$ could be spin up and $\sigma = -1$ could be spin down. We can also use this to represent a lattice, so +1 means the spot is occupied by a particle and -1 means it is not. The point is the Ising model is very useful when it comes to describing physical systems.

First we want to write down the Hamiltonian for this system, which is a function that represents the energy of every possible configuration (in physics speak these are microstates).

So the general Hamiltonian for the Ising Model is:

$$\mathcal{H} = -\frac{J}{2}\sum_{\langle i,j\rangle} \sigma_i \sigma_j - B\sum_i \sigma_i \,, \quad \sigma_i = \pm 1 \qquad \text{(Eq 1.2.1)}$$

The first term is the **interaction term**, between neighboring spins. The coefficient tells you how strongly the neighboring spins are coupled, and the sign tells you whether they prefer to align or anti-align. For magnets, this would tell you whether they behave as ferromagnets or antiferromagnets. Also for the interaction term in the Ising Model, it is necessary to note a few simplifications.

1. We are assuming that each site only interacts with the site directly adjacent to it.
2. The spins are scalar quantities rather than vectors. So this is the +1 or -1 that we discussed before.

The second term is the **external field** term. B in physics usually refers to the magnetic field, so its magnitude will tell you how strong the external field is, and the sign will tell you if spin up or spin down is preferred.

Once we have the Hamiltonian, we want to use it to find thermodynamic properties of our magnet or other substance.

To do this, we first diagonalize the hamiltonian to find a basis of energy states. The we calculate the partition function, which in last class we defined to be

$$Z(N,\beta) = \sum_{C} e^{-\beta E(C)}.$$

Where

$E(\mathcal{C})$ is the energy of the configuration $\mathcal{C}$ while T is the absolute temperature. .

and

$$\beta = 1/kT.$$

This can be rewritten as:

$$Z(N,\beta) = \sum_{\mathcal{C}} e^{-\beta E(\mathcal{C})} = \sum_{E} \omega(E)\, e^{-\beta E} = \sum_{E} e^{-\beta E + \log \omega(E)} =$$
$$= \sum_{E} e^{\beta[TS-E]} \equiv e^{-\beta F(N,\beta)},$$

Where $F(N,\beta)$ is the *free energy* of the system.

This free energy is not so important for the purposes of this class, I am more mentioning it to motivate some of the physical applications behind the partitions function.

So with these quantities, we can compute the expectation values for any physical observable:

$$\langle \mathcal{O} \rangle = Z^{-1} \sum_{\mathcal{C}} \mathcal{O}(\mathcal{C})\, e^{-\beta E(\mathcal{C})}.$$

Now let's get to actually solving the 1D Ising Model. In this talk, I will present two different methods for solving the model. The first is called the recursive method.

The Recursive Method

In this method, we will consider a linear chain of N particles with some spin, either spin up or spin down, in the absence of an external magnetic field and with free boundary conditions on the first and last spin of the chain.

So visually it looks like this:

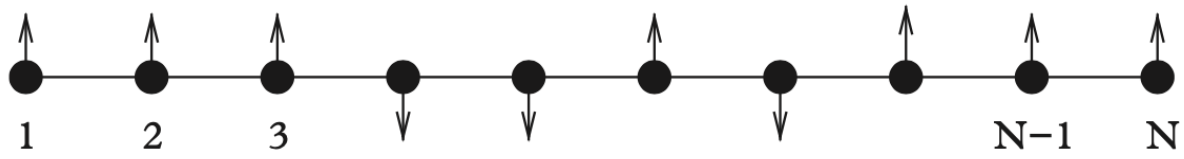


Fig. 2.1 *Linear chain of N Ising spins.*

Since our model has no external magnetic field, we can neglect the second term of the Hamiltonian, and it simply becomes:

$$\mathcal{H} = -\sum_{i=1}^{N-1} \mathcal{J}_i \sigma_i \sigma_{i+1}$$

Plugging this into our equation for Z, we get the resulting double summation for our partition function:

$$\mathcal{H} = -\sum_{i=1}^{N-1} \mathcal{J}_i \sigma_i \sigma_{i+1} \qquad Z(N,\beta) = \sum_{C} e^{-\beta E(C)}$$

$$J_i = \beta \mathcal{J}_i$$

$$Z_N = \sum_{\sigma_1=-1}^{1} \sum_{\sigma_2=-1}^{1} \cdots \sum_{\sigma_N=-1}^{1} \exp\left(\sum_{i=1}^{N-1} \mathcal{J}_i \sigma_i \sigma_{i+1}\right),$$

The pluses and minuses here are based on the exact model I drew.

The recursive method approach involves adding an extra spin to the chain and expressing the resulting partition function Z_{N+1} in terms of the previous Z_N

So by adding another spin to the sum, we get the following:

$$Z_{N+1} = \sum_{\sigma_1=-1}^{1} \sum_{\sigma_2=-1}^{1} \cdots \sum_{\sigma_N=-1}^{1} \exp\left(\sum_{i=1}^{N-1} \mathcal{J}_i \sigma_i \sigma_{i+1}\right) \sum_{\sigma_{N+1}=-1}^{1} \exp\left(\mathcal{J}_N \sigma_N \sigma_{N+1}\right). \quad (2.1.2)$$

The last sum can be computed using our definition of hyperbolic trig functions.

$$\sum_{\sigma_{N+1}=-1}^{1} \exp\left(\mathcal{J}_N \sigma_N \sigma_{N+1}\right) = e^{\mathcal{J}_N \sigma_N} + e^{-\mathcal{J}_N \sigma_N} = 2\cosh(\mathcal{J}_N \sigma_N) = 2\cosh \mathcal{J}_N,$$

Recall: $$\cosh x = \frac{e^x + e^{-x}}{2}$$

As you may notice, the result of this final summation is independent of σ_N. That is to say it is independent of the state of the particle we just added on.

So this allows us to rewrite the partition function as:

$$Z_{N+1} = (2\cosh \mathcal{J}_N) Z_N,$$

And the iteration in terms of Z_1 is:

$$Z_{N+1} = \left(2^N \prod_{i=1}^{N} \cosh \mathcal{J}_i\right) Z_1.$$

Since the partition function Z_1 of a single spin is equal to the number of possible states, Z_1 just equals 2.

We can therefore write the exact expression for the partition function of *N* spins as:

$$Z_N = 2^N \prod_{i=1}^{N-1} \cosh \mathcal{J}_i.$$

So now that we have solved for the partition function, we can use this to find observables such as the two-spin correlation function, which tells us how the spin at one site influences the spin at another site.

<u>Two Spin Correlation Calculation (Recursive Method)</u>

So we know the expression for the partition function and the expression for the two spin correlation:

know: $Z_N = \sum_{\{\theta\}} \prod_{i=1}^{N} e^{J_i \sigma_i \sigma_{i+1}}$ and $G_k^{(2)}(1) = Z_N^{-1} \sum_{\{\theta\}} \sigma_k \sigma_{k+1} \prod_{i=1}^{N-1} e^{J_i \sigma_i \sigma_{i+1}}$

$$Z_N G_k^{(2)}(1) = \sum_{\{\theta\}} \sigma_k \sigma_{k+1} \prod_{i=1}^{N-1} e^{J_i \sigma_i \sigma_{i+1}}$$

To find $G_k^{(2)}(r)$, let us first take r=1 as an example.

If we take the partial derivative of the partition function with respect to J_k, we get (1).

And then if we take a second partial derivative, moving each index down one spot, (2)

We noticed that these two expressions are the same (1) and (2). So as you keep taking partial derivatives of Z_N, you actually get out the previous partial derivative.

Therefore, we can replace the partial derivative in (2) with our expression for the partial derivative of Z_N, and we get (3).

We can actually further simplify this by recognizing that when two σ's with the same index are multiplied, that is just σ^2, which equals 1. Because σ can only be +/- 1.

Therefore, since this expression in (3) is just Z, we can write (4).

Then substituting this into the expression for Z_N and using our hyperbolic trig identities, we get (5)

$$\frac{\partial}{\partial J_k} Z_N = \sum_{\{\theta\}} \frac{\partial}{\partial J_k} \left(\prod_{i=1}^{N-1} e^{J_i \sigma_i \sigma_{i+1}} \right) \quad (1)$$

$$= \sum_{\{\theta\}} \sigma_k \sigma_{k+1} \prod_{i=1}^{N-1} e^{J_i \sigma_i \sigma_{i+1}}$$

same

$$\frac{\partial}{\partial J_k} \left(\frac{\partial}{\partial J_{k+1}} Z_N \right) = \frac{\partial}{\partial J_k} \left(\sum_{\{\theta\}} \sigma_{k+1} \sigma_{k+2} \prod_{i=1}^{N-1} e^{J_i \sigma_i \sigma_{i+1}} \right)$$

$$= \sum_{\{\theta\}} \sigma_{k+1} \sigma_{k+2} \frac{\partial}{\partial J_k} \left(\prod_{i=1}^{N-1} e^{J_i \sigma_i \sigma_{i+1}} \right) \quad (2)$$

$$= \sum_{\{\theta\}} \sigma_{k+1} \sigma_{k+2} \sigma_k \sigma_{k+1} \left(\prod_{i=1}^{N-1} e^{J_i \sigma_i \sigma_{i+1}} \right) \quad (3)$$

$\sigma = \pm 1 \Rightarrow \sigma_k^2 = 1$; $\sigma_{k+1}\sigma_{k+2}\sigma_k\sigma_{k+1} = 1$; $\prod_{i=1}^{N-1} e^{J_i \sigma_i \sigma_{i+1}} = Z_N$

$$\therefore Z_N G^{(2)}(r) = \frac{\partial}{\partial J_k} \frac{\partial}{\partial J_{k+1}} \cdots \frac{\partial}{\partial J_{k+r-1}} Z_N \quad \text{for arbitrary } r \quad (4)$$

sub into $Z_N = 2^N \prod_{i=1}^{N-1} \cosh J_i$

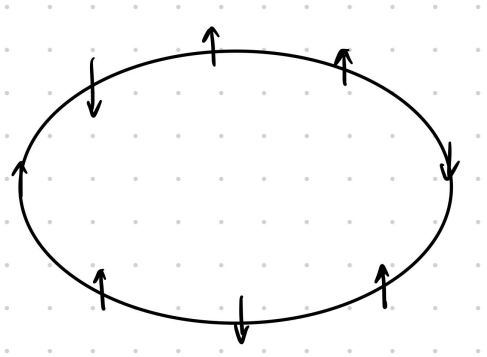
$$\boxed{G^{(2)}(r) = \prod_{i=1}^{r} \tanh J_{k+i-1}} \quad \text{two spin correlation} \quad (5)$$

So this is one method of solving for the partition function. But this method has some limitations, namely that it does not take into account an external magnetic field. The next method that I am going to present can be applied in the presence of an external magnetic field, and also has many parallels to the Feynman path integral in Quantum Mechanics.

The Transfer Matrix Method

Once again taking the one dimensional case, this time we are going to consider a chain of spins in the homogenous case, which just means that there is only one coupling constant.

So our model looks something like this:



And we will first analyze the periodic boundary condition, where

$$\sigma_i \equiv \sigma_{N+i}.$$

So essentially every *N* spins the pattern repeats.

The Hamiltonian for this setup will be:

$$\mathcal{H} = -\mathcal{J} \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1} - B \sum_{i=1}^{N} \sigma_i.$$

And thus the partition function is:

$$Z_N(N,\beta) = \sum_{\sigma_1} \sum_{\sigma_2} \cdots \sum_{\sigma_N} \exp\left[-\beta\left(-\mathcal{J} \sum_{i=1}^{N} \sigma_i \sigma_{i+1} - B \sum_{i=1}^{N} \sigma_i\right)\right]$$

$$\sigma_i = \pm 1$$

And then the textbook makes the following substitutions,

And then the textbook makes the following substitutions:

sub: $J = \beta J$ and $B = \beta B$ $\quad \sigma_i = \sigma$ $\quad \sigma_{i+1} = \sigma'$

so we can write:

$$Z_N = \sum_{\sigma_1} \sum_{\sigma_2} \cdots \sum_{\sigma_N} \exp\left[J \sum_{i=1}^{N} \sigma\sigma' + B \sum_{i=1}^{N} \{\sigma + \sigma'\} \right]$$

So just to preface this derivation, when I was working it out myself, I did it the way I learned in Quantum Mechanics, which is using the bra-ket notation and the completeness relation. I am going to show that method first since it makes the most intuitive sense to me, and then I'm going to do it another way if you are not familiar with this method.

Ok so we can combine the two summations above to get the following for Z_N:

$$Z_N = \sum_{\sigma_1} \sum_{\sigma_2} \cdots \sum_{\sigma_N} \exp\left[J\sigma\sigma' + \frac{1}{2} B(\sigma + \sigma') \right]$$

So where the Quantum Mechanics comes in here:

We want to define a 2×2 matrix V with elements:

$$\langle \sigma | V | \sigma' \rangle = e^{[J\sigma\sigma' + \frac{1}{2}B(\sigma+\sigma')]}$$

where σ and σ' independently take on either $+1$ or -1.

Matrix:

$$\begin{pmatrix} \langle +1|V|+1\rangle & \langle -1|V|+1\rangle \\ \langle +1|V|-1\rangle & \langle -1|V|-1\rangle \end{pmatrix} \qquad \begin{matrix} 11 & 12 \\ 21 & 22 \end{matrix}$$

First row + column → σ

Plug in ± 1 to

We get:

$$\begin{pmatrix} e^{J+B} & e^{-J} \\ e^{-J} & e^{J-B} \end{pmatrix}$$

We can then write the partition function in the form:

$$Z_N = \sum_{\sigma_1}\sum_{\sigma_2}\cdots\sum_{\sigma_N} \langle \sigma_1|V|\sigma_2\rangle\langle \sigma_2|V|\sigma_3\rangle \cdots \langle \sigma_N|V|\sigma_1\rangle$$

$\langle \sigma_2|V|\sigma_2\rangle = 1$

Can get rid of middle terms using completeness relation:

$$\sum_{\sigma=\pm 1} |\sigma\rangle\langle\sigma| = 1$$

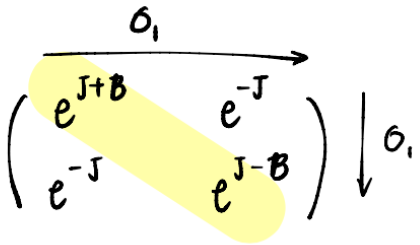
becomes:

$$Z_N = \sum_{\sigma_1}\langle \sigma_1|V\left[\sum|\sigma_2\rangle\langle\sigma_2|\right]V\left[\sum_{\sigma_3}|\sigma_3\rangle\langle\sigma_3|\right]V \cdots V\left[\sum_{\sigma_N}|\sigma_N\rangle\langle\sigma_N|\right]V|\sigma_1\rangle$$

$$= 1 \qquad = 1 \qquad = 1$$

$$Z_N = \sum_{\sigma_1}^{1}\langle \sigma_1|V^N|\sigma_1\rangle$$

$$= \mathrm{Tr}(V^N) \qquad \text{definition of a trace}$$



(def of a trace → indices are the same)

Ok so that was the way that I did it, but when I was working this out with Cailan we also did it another way, which is shown here:

Another proof: $Z_N = \mathrm{Tr}(V^N)$

Definition: $V(\sigma_i, \sigma_j) = e^{J\sigma_i\sigma_j + \frac{1}{2}B(\sigma_i + \sigma_j)}$

We know eq for Z_N to be:

$$Z_N = \sum_{\{\sigma\}} e^{\sum_{i=1}^{N} J\sigma_i\sigma_{i+1} + \sum_{i=1}^{N} B\sigma_i} = \sum_{\{\sigma\}} \prod_{i=1}^{N} e^{J\sigma_i\sigma_{i+1} + B\sigma_i}$$

combine sums

use product operator

⇓ write out term by term

$$= \sum_{\{\sigma\}} e^{J\sigma_1\sigma_2 + B\sigma_1} e^{J\sigma_2\sigma_3 + B\sigma_2} \cdots e^{J\sigma_N\sigma_1 + B\sigma_1}$$

equivalent of $V(\sigma_1, \sigma_2)V(\sigma_2, \sigma_3) \cdots V(\sigma_N, \sigma_1)$

Show:

$$V(\sigma_1, \sigma_2)V(\sigma_2, \sigma_3)V(\sigma_N, \sigma_1) = e^{J\sigma_1\sigma_2 + \frac{1}{2}B(\sigma_1 + \sigma_2)} e^{J\sigma_2\sigma_3 + \frac{1}{2}B(\sigma_2 + \sigma_3)} e^{J\sigma_N\sigma_1 + \frac{1}{2}B(\sigma_N + \sigma_1)} =$$

multiply exponentials ⇒ adding their powers

each has power of $\frac{1}{2}B\sigma$

can combine to get: $e^{J\sigma_1\sigma_2 + B\sigma_1} e^{J\sigma_2\sigma_3 + B\sigma_2} \cdots e^{J\sigma_N\sigma_1 + B\sigma_N}$

Therefore:

$$\sum_{\{\sigma\}} V(\sigma_1,\sigma_2)V(\sigma_2,\sigma_3)\cdots V(\sigma_N,\sigma_1) = \sum_{\{\sigma\}} e^{J\sigma_1\sigma_2+B\sigma_1}\, e^{J\sigma_2\sigma_3+B\sigma_2} \cdots e^{J\sigma_N\sigma_1+B\sigma_1} = Z_N$$

Now write V as matrix:

$$V = \begin{pmatrix} e^{J+B} & e^{-J} \\ e^{-J} & e^{J-B} \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$$

Say N=2:

$$\sum_{\{\sigma\}} V(\sigma_1,\sigma_2)V(\sigma_2,\sigma_1) = \sum V_{12}V_{21}$$

$$V^2 = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} = \begin{pmatrix} V_{11}V_{11}+V_{12}V_{21} & V_{11}V_{12}+V_{12}V_{22} \\ V_{21}V_{11}+V_{22}V_{21} & V_{21}V_{12}+V_{22}V_{22} \end{pmatrix}$$

$Tr(V^2) = V_{11}V_{11} + V_{12}V_{21} + V_{21}V_{12} + V_{22}V_{22}$ inner indices are the same

$$Tr(V^2) = \sum_{i,j=1}^{2} V_{ij}V_{ji}$$

It follows that $Z_N = Tr(V^N)$

This is essentially the completeness relation expressed in matrix form.

Now, let us solve for the eigenvalues. Again, there are multiple ways to do this. I am going to show how I did it first, which is just using the eigenvalue equation, but the book does it using the unitary matrix, which will come in handy later, so I am going to show that as well.

So, first the method using the eigenvalue equation:

Recall: The trace of a matrix is equal to the sum of its eigenvalues: $Tr(V^N) = \sum_{i=1}^{N} \lambda_i^N$

$$Z = Tr\left[\begin{pmatrix} e^{J+B} & e^{-J} \\ e^{-J} & e^{J-B} \end{pmatrix}^N\right] = \lambda_+^N + \lambda_-^N$$

Recall eigenvalue equation:

$$\vec{A}\vec{v} = \lambda\vec{v}$$

$$(\vec{A} - \lambda\mathbb{1})\vec{v} = 0$$

$$\det(\vec{A} - \lambda\mathbb{1}) = 0$$

Plug in V: $$\det\begin{pmatrix} e^{J+B} & e^{-J} \\ e^{-J} & e^{J-B} \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

$$(e^{J+B} - \lambda)(e^{J-B} - \lambda) - e^{-2J} = 0$$

$$\underbrace{(e^{2J} - e^{-2J})} - \lambda e^{J}(e^{B} + e^{-B}) + \lambda^2 = 0$$

$$2\sinh(2J) - \lambda e^{J} 2\cosh(B) + \lambda^2 = 0$$

Use Quadratic formula to solve:

$$\lambda_\pm = e^J \cosh B \pm \sqrt{e^{2J}\cosh^2 B - 2\sinh(2J)}$$

simplify: $\cosh^2 x = 1 + \sinh^2(x)$

OR

$$\cosh^2 x - \sinh^2(x) = 1$$

$$\lambda = e^J \cosh(B) \pm \sqrt{e^{2J}(1+\sinh^2 B) - (e^{2J} - e^{-2J})}$$

$$\lambda = e^J \cosh(B) \pm \sqrt{e^{2J}\sinh^2 B + e^{-2J}}$$

Next I will show how the book did it using the Unitary Matrix to diagonalize V.

V is Hermitian (self adjoint, is its own transpose conjugate)

U = unitary matrix

can use U to diagonalize V:

$$U^{-1} V U = \mathcal{O} = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$$

can use 2D rotation matrix as U: $\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$ Define: $\cot(2\phi) = e^{2J}\sinh B$

we are essentially rotating to the basis where V is diagonal

Insert identity matrix $\mathbb{1}$ in form of $UU^{-1} = \mathbb{1}$

$$\mathrm{Tr}(V^N) = \mathrm{Tr}[UU^{-1}V^N] = \mathrm{Tr}[U^{-1}V^N U] = \mathrm{Tr}(\mathcal{O}^N) = \lambda_+^N + \lambda_-^N$$

get same eigenvalues as above

Again, like in the recursive method, we can use this to find the two spin correlation function.

So we have the expression for the two spin correlation:

$$\langle \sigma_1 \sigma_{r+1} \rangle = Z_N^{-1} \sum_{\{\sigma\}} \sigma_1 V(\sigma_1, \sigma_2) \cdots \sigma_{r+1} V(\sigma_{r+1}, \sigma_{r+2}) \cdots V(\sigma_N, \sigma_1)$$

Want to represent this expression in matrix form

To do that, introduce Herminan or self adj matrix S

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

say $V^r = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Can use SV^rS to get V^r into basis states of σ_1 and σ_{r+1}

$$SAS = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & b \\ -c & -d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

This expression: $Z_N^{-1} \sum_{\{\sigma\}} \sigma_1 V(\sigma_1, \sigma_2) \cdots \sigma_{r+1} V(\sigma_{r+1}, \sigma_{r+2}) \cdots V(\sigma_N, \sigma_1)$ — expression represented by matrix

$$= \begin{cases} V_{11}^r & \text{when } \sigma_1 = 1, \sigma_{r+1} = 1 \\ -V_{12}^r & \text{when } \sigma_1 = 1, \sigma_{r+1} = -1 \\ -V_{21}^r & \text{when } \sigma_1 = -1, \sigma_{r+1} = 1 \\ V_{22}^r & \text{when } \sigma_1 = -1, \sigma_{r+1} = -1 \end{cases}$$

Then: $\langle \sigma_1 \sigma_{r+1} \rangle = Z_N^{-1} \sum_{\{\sigma\}} SV^rS(\sigma_1, \sigma_{r+1}) \underbrace{V(\sigma_{r+1}, \sigma_{r+2}) \cdots V(\sigma_N \sigma_1)}_{V^{N-r}}$

$$= Z_N^{-1} \operatorname{Tr}(SV^rSV^{N-r})$$

$$\mathrm{Tr}(SV^rSV^{N-r}) = \mathrm{Tr}(UU^{-1}SV^rSV^{N-r})$$

$$= \mathrm{Tr}(U^{-1}SUU^{-1}V^rSV^{N-r}U)$$

set $U^{-1}SU = A$

$$= \mathrm{Tr}(\underbrace{AU^{-1}V^rUU^{-1}}_{\mathcal{O}}\underbrace{SUU^{-1}}_{A}\underbrace{V^{N-r}U}_{\mathcal{O}^{N-r}})$$

Insert another set of unitary matrices

$\mathcal{O}$ = Diagonalized V^r

$$= \mathrm{Tr}(A\mathcal{O}^rA\mathcal{O}^{N-r})$$

$$A\mathcal{O}^rA\mathcal{O}^{N-r} = \begin{pmatrix}\cos 2\phi & -\sin 2\phi \\ -\sin 2\phi & -\cos 2\phi\end{pmatrix}\begin{pmatrix}\lambda_+^r & 0 \\ 0 & \lambda_-^r\end{pmatrix}\begin{pmatrix}\cos 2\phi & -\sin 2\phi \\ -\sin 2\phi & -\cos 2\phi\end{pmatrix}\begin{pmatrix}\lambda_+^{N-r} & 0 \\ 0 & \lambda_-^{N-r}\end{pmatrix}$$

$$= \begin{pmatrix}\lambda_+^r\cos 2\phi + \lambda_-^r\sin^2 2\phi & 0 \\ 0 & \lambda_+^r\sin^2 2\phi\ \lambda_-^r\cos^2 2\phi\end{pmatrix}\begin{pmatrix}\lambda_+^{N-r} & 0 \\ 0 & \lambda_-^{N-r}\end{pmatrix}$$

$$= \begin{pmatrix}\lambda_+^N\cos^2\phi + \lambda_+^N\left(\frac{\lambda_-}{\lambda_+}\right)\sin^2 2\phi & 0 \\ 0 & \lambda_-^N\left(\frac{\lambda_+}{\lambda_-}\right)^r\sin^2 2\phi + \lambda_-^N\cos^2 2\phi\end{pmatrix}$$

Take limit as $N\to\infty$, $\left(\frac{\lambda_-}{\lambda_+}\right)^N \to 0$

$$\langle G_1, G_{r+1}\rangle = \frac{\mathrm{Tr}(A\mathcal{O}^rA\mathcal{O}^{N-r})}{\mathrm{Tr}(\mathcal{O}^N)} = \frac{\lambda_+^N\cos^2\phi + \lambda_+^N\left(\frac{\lambda_-}{\lambda_+}\right)\sin^2 2\phi}{\lambda_+^N}$$

$$\boxed{\langle G_r, G_{r+1}\rangle = \cos^2 2\phi + \left(\frac{\lambda_-}{\lambda_+}\right)^r\sin^2 2\phi}$$

Other Boundary Conditions

We previously analyzed a periodic boundary condition, where we started at some state and then eventually came back to that same state. Now we are going to look at instances where the beginning and ending boundary conditions are distinct. This is particularly useful in Quantum Mechanics, where we have an initial and final state. And to compute the partition function, we simply need to evaluate the matrix element of the quantum time evolution operator between the initial state and the final state.

So let's set the initial state <a| and final state |b>.

Now, we have the partition function

$$Z_N^{(a,b)} = \sum_{\sigma_2 = \pm 1} \cdots \sum_{\sigma_{N-1} = \pm 1} \langle a|V|\sigma_2\rangle\langle\sigma_2|V|\sigma_3\rangle \cdots \langle\sigma_{N-1}|V|b\rangle =$$

$$= \langle a|V^{N-1}|b\rangle$$

Notice: <a| takes the place of σ_1, so the sums start at σ_2, and b takes the place of σ_N, so V is taken to the power of N – 1 as opposed to N.

Then,, we use the same techniques as before to solve for Z, by diagonalizing V^{N-1} using the unitary matrix U.

$$Z^{(a,b)} = \langle a | U U^{-1} V^{N-1} U U^{-1} | b \rangle$$

$$= \langle a | U \mathcal{O}^{N-1} U^{-1} | b \rangle$$

Now, we can look at some explicit examples of initial and final states.

If we have the initial and final state as spin up, or 1.

$$|a\rangle = |b\rangle = |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We can then compute the matrix element using the expression above and find the partition function.

$$Z_N^{++} = \quad \langle + \mid U \mathcal{D}^{N-1} U^{-1} \mid + \rangle =$$

$$= (1,0) \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \lambda_+^{N-1} & 0 \\ 0 & \lambda_-^{N-1} \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} =$$

$$= \quad \lambda_+^{N-1} \cos^2\phi + \lambda_-^{N-1} \sin^2\phi.$$

Further, if we have a free boundary condition,

$$|\,\mathrm{f}\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

we can perform a similar calculation to the one above only replacing the vectors on the end with (1 1) to get:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \lambda_+^{N-1} & 0 \\ 0 & \lambda_-^{N-1} \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$Z_N^{ff} = Z_N^{++} + Z_N^{--} + 2Z_N^{+-} = \lambda_+^{N-1} + \lambda_-^{N-1} + \sin 2\phi \left(\lambda_+^{N-1} - \lambda_-^{N-1}\right).$$

So this is all I have for my talk about the Ising Model. Again, this is a mathematical model, but it has a lot of physical applications to understanding how matter transitions in between different phase states.