## Part A - Circulant Matrices

## Intro to Circulant Matrices

Broad idea: A circulant matrix is an $n \times n$ matrix where all row vectors are composed of the same elements and each row vector is rotated one element to the right relative to the preceding row vector also called a circular shift. These rows "wrap" around the edges cyclically which gives the name circulant. Alternatively, the matrix can be viewed as a the columns being shifted down.

$$
C=\left(\begin{array}{ccccc}
c_{0} & c_{1} & c_{2} & \cdots & c_{n-1} \\
c_{n-1} & c_{0} & c_{1} & c_{2} & \cdots \\
c_{n-2} & c_{n-1} & c_{1} & \cdots & \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
c_{1} & c_{2} & \cdots & c_{n-1} & c_{0}
\end{array}\right)
$$

An example of a circulant matrix is:
and a generalized example for $n=4$ :

$$
C_{4}=\left(\begin{array}{llll}
c_{0} & c_{1} & c_{2} & c_{3} \\
c_{3} & c_{0} & c_{1} & c_{2} \\
c_{2} & c_{3} & c_{0} & c_{1} \\
c_{1} & c_{2} & c_{3} & c_{0}
\end{array}\right)
$$

All Circulant matrices are self-adjoint that is a matrix that is equal to its own conjugate transpose (elements at position ij equal the complex conjugate of the elements at $j i$ ).
This means all of its eigenvalues are real.
They subset the Toeplitz matrix which is often used to compute a discrete convolution by converting one input into a Toeplitz matrix. These are of the form:
$T_{4}=\left(\begin{array}{llll}c_{0} & c_{1} & c_{2} & c_{3} \\ c_{4} & c_{0} & c_{1} & c_{2} \\ c_{5} & c_{4} & c_{0} & c_{1} \\ c_{6} & c_{5} & c_{4} & c_{0}\end{array}\right)$
This additional structural requirement is beneficial because they have orthonormal eigenvectors and we know exactly what their eigenvectors are. (Complex-Symmetric)

Their eigenvectors are also closely related to the Fourier Transform and Fourier Series.
They are used by computer algorithms to compute a Fast Fourier Transform (FFT) or a Discrete Fourier Transform(DFT) since a true Fourier Series is often to slow to compute practically.

## Multiplying Circulant Matrices: Convolutions

To multiply a Circulant matrix start with standard matrix multiplication:
$y=C x=\left(\begin{array}{ccccc}c_{0} & c_{1} & c_{2} & \cdots & c_{n-1} \\ c_{n-1} & c_{0} & c_{1} & c_{2} & \cdots \\ c_{n-2} & c_{n-1} & c_{0} & \cdots & \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ c_{1} & c_{2} & \cdots & c_{n-1} & c_{0}\end{array}\right)\left(\begin{array}{c}x_{0} \\ x_{1} \\ \vdots \\ x_{n-1}\end{array}\right)$
$y=\left(\begin{array}{c}c_{0} x_{0}+c_{1} x_{1}+c_{2} x_{2} \ldots \\ c_{n-1} x_{0}+c_{0} x_{1}+c_{1} x_{2} \ldots \\ c_{n-2} x_{0}+c_{n-1} x_{1}+c_{0} x_{2} \ldots \\ \vdots \\ c_{1} x_{0}+c_{2} x_{1}+c_{3} x_{2} \ldots\end{array}\right)$ utilizing the pattern here we can rewrite any row of
$y$ as: $y_{k}=\sum_{j=0}^{n} c_{j-k} x_{j}$ to view it easier consider the "rows" of $y$ as $k$ and the terms or "columns" as $j$.
Note there is a problem with the $(j-k)$ subscript. When $j<k$ we get a negative index however because of the matrix's circulant nature we can view the subscript as $n$ periodic which means we can take any $(j-k) \bmod n$ to get the equivalent index.

For example: Look at the first term in the last row in the matrix that is $k=n-1$ (because we started our index at 0 ) and $j=0$ then:
$(j-k) \bmod n=(0-(n-1) \bmod n=-n \bmod n+1 \bmod n=0+1=1$ which equals our expected $c_{1}$ index.

This multiplication process is also called a circular convolution which is a special case of a periodic convolution. Periodic convolution arises in the discrete-time Fourier transform, Cryptography(the mixColumns step of AES), signal processing and so on.

## Eigenvectors of Circulant matrices

One property of circulant matrices is the eigenvectors are always the same despite the eigenvalues being different for each $C$. Since we know the eigenvectors the matrices are easy to diagonalize. One of the eigenvalues is easy to calculate $\lambda_{0}$.

Take eigenvector: $x^{(0)}=\left(\begin{array}{c}1 \\ 1 \\ \vdots \\ 1\end{array}\right)$ this is an eigenvector because multiplying $C x^{(0)}$ simply sums each row of $C$ but because each row contains the same entries in a different order the sum is the same so we have: $C x^{(0)}=\left(c_{0}+\ldots+c_{n-1}\right) x^{(0)}$ where $\lambda_{0}=$ $c_{0}+c_{1}+\ldots+c_{n-1}$

The other Eigenvectors are found by writing them down in terms of a primitive root of unity: $\omega_{n}=e^{\frac{2 \pi i}{n}}$ where $\omega_{n}^{n}=e^{2 \pi i}=1=\omega_{n}^{0}$ but no smaller power equals 1 and therefore $\omega_{n}^{j+n}=\omega_{n}^{j} \omega_{n}^{n}=\omega_{n}^{j}$ meaning the exponents of $\omega_{n}$ are $n$ periodic and $n$ is prime.

Note: This is the same periodicity we used to show that $c_{j-k}$ entry was really $(j-k)$ $\bmod n$.

This forms the remaining eigenvectors where the $n$-th roots of unity construct $n$ vectors minus the "original" one already computed.

For example take $n=7$ then we have 7 roots of unity represented by:

They are the "roots of unity" because $\omega_{n}^{j}$ for $j=0, \ldots, n-1$ are all solutions to $z^{n}=1$ where $z$ is a complex number satisfying the equation.


See Handwritten notes for examples of how to compute this first page.

Eigenvectors to the DFT:
In terms of $\omega_{n}$, the eigenvectors of a circulant matrix are the $k$-th eigenvector $x^{(k)}$ for $k=0, \ldots, \mathrm{n}-1$ for any $n \times n$ circulant matrix is $x^{(k)}=\left(\begin{array}{c}\omega_{n}^{0 k} \\ \omega_{n}^{1 k} \\ \vdots \\ \omega_{n}^{(n-1) k}\end{array}\right)$ then constructing a matrix $F$ whose columns are the eigenvectors is: $F=\left[\begin{array}{llll}x^{(0)} & x^{(1)} & \ldots & x^{(n-1)}\end{array}\right]$ with entries $F_{j k}=x_{j}^{(k)}=\omega_{n}^{j k}=e^{\frac{2 \pi i}{n j k}}$ here the $k$ corresponds to columns and the $j$ corresponds to rows which is opposite from how $y_{k}$ was represented earlier.

Multiplying a vector by $F$ performs a DFT which is a finite version to computing a Fourier series.

Some examples of $F$ are: $F_{2}=\left(\begin{array}{cc}1 & 1 \\ 1 & -1\end{array}\right)$ and $F_{4}=\left(\begin{array}{cccc}1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i\end{array}\right)$ Or more generally $F_{4}=\left(\begin{array}{cccc}1 & 1 & 1 & 1 \\ 1 & i & i^{2} & i^{3} \\ 1 & i^{2} & i^{4} & i^{6} \\ 1 & i^{3} & i^{6} & i^{9}\end{array}\right)$ to make it unitary multiply the matrix by $\frac{1}{\sqrt{n}}$
Then if we took some matrix $A$ that is circular we could calculate $F^{-1} A F=A_{D}$ where $A$ has been diagonalized and the diagonal entries are equal to the eigenvalues of $A$.

Note: Even if $A$ is real-symmetric, it may not be the case that the eigenvectors are all real but rather they could chosen to be real as Eigenvalues tend to come in pairs where these real or imaginary eigenvalues correspond to the discrete cosine transform (DCT) and discrete sine transform (DST).

To see why this works take the formula for $C x$ and multiply it by an eigenvector.
Let $y_{l}=C x^{(k)}$ then an $l$-th component is: $y_{l}=\sum_{j=0}^{n-1} c_{j-l} \omega_{n}^{l k}=\omega_{n}^{l k} \sum_{j=0}^{n-1} c_{j-l} \omega_{n}^{(j-l) k}$
but the remaining sum is now independent of $l$ because both $c_{j}$ and $\omega_{n}^{j}$ are periodic in $j$, all $j \Longrightarrow j-l$ does is rearrange the number being summed or circular shifts them
so the sum is same and $\omega_{n}^{l k}=x^{(k)}$ yielding: $C x^{(k)}=\lambda_{k} x^{(k)}$ where $\lambda_{k}=\sum_{j=0}^{n-1} c_{j} \omega_{n}^{j k}$ but if we define a vector $\hat{c}=\left(\lambda_{0}, \lambda_{1}, \ldots, \lambda_{n-1}\right)$ then $\hat{c}=F c$ where $c=$ first row of $C$ this shows the eigenvalues are the DFT of $c$

## Definitions:

Periodic Convolution:
A convolution of two periodic function that have the same period. Given two $T$-period functions $h_{T}(t)$ and $x_{T}(t)$ the equation for convolution is $\int_{t_{0}}^{t_{0}+T} h_{T}(\tau) \cdot x_{T}(t-\tau) d \tau$

Real Symmetric:
A real $n \times n$ matrix $A$ is symmetric iff $\langle A x, y\rangle=\langle x, A y\rangle, \forall x, y \in \mathbb{R}$ for the standard inner product.

Positive Semidefinite:
Let $A$ be a symmetric matrix and $Q(x)=x^{T} A x$ the corresponding quadratic form, then $Q$ and $A$ are called positive definite if $Q(x)>0, \forall x \neq 0$. So positive semidefinite means there are no minuses in the signature, while positive definite me

Dimer Solutions
Recall $\sum d(m, n)_{r, s} x^{r} y^{s}=\operatorname{Pf}(A(m, n))$ whese $d(m, n)_{r, s}$ $=$ \# of cimess on $m \times n$ lattice with $r$ horizontal links


Readil: $A=\left[\begin{array}{ll}a_{11} & a_{12} \\ a_{n} & \ddots\end{array}\right], A \otimes B=\left[\begin{array}{ll}a_{11} B & a_{12} B \\ a_{21} B & \ddots\end{array}\right]$ let $H_{k}=\left[\begin{array}{ccc}0 & 1 & \\ -10 & 1 & \\ -10 & 1 & 1 \\ & -1 & 0\end{array}\right] \quad Z_{k}=\left[\begin{array}{cc}-1 & \\ 1 & \\ & -1\end{array}\right]$ then $I_{3} \otimes x H_{2}=\left[\begin{array}{cc|c|c}0 & x & & \\ -x & 0 & & \\ \hline & 0 & x & \\ \hline & & 0 & \\ \hline & & & -x \\ & & x & 0\end{array}\right]$ and $y H_{3} \otimes z_{2}=\left[\begin{array}{l|l|l}-y_{4} \mid \\ \hline y & -y \\ -y & & y \\ \hline & y & -y\end{array}\right]$

$$
A(2,3)=I_{3} \otimes H_{2}+y H_{3} \otimes z_{2} \Rightarrow A(m, n)=I_{n} \otimes \times H_{m}+y H_{n} \otimes z_{m}
$$

THM $H_{K}$ is diagnalizable $w /$ transtion matrix $S_{K}\left(S_{K}^{-1} H_{K} S_{K}=D_{K}\right)$ eigenvalues: $\quad \lambda_{k}(b)=Z_{i} \cos \left(\frac{\pi b}{k+1}\right) 1 \leq b \leq k$


$$
\begin{aligned}
& \text { irculant } \\
& \Rightarrow e v^{\prime} s \text { ade } A^{\prime}(b)=1 e^{\frac{2 \pi i b}{k}}-1 e^{\frac{2 \pi i(k-1)}{k}} \\
& =e^{\frac{2 \pi i b}{k}}-e^{\frac{-2 \pi i b}{k}}=2_{i} \sin \left(\frac{2 \pi b}{k}\right)
\end{aligned}
$$

Lemma: $S_{k}^{-1} Z_{k} S_{k}=X_{k}=\left[\begin{array}{cc}0 & 1 \\ 1 & 0\end{array}\right]$ Fact $:(A \otimes B)(C \otimes D)=A C \otimes B D$ Hoad: compute $\operatorname{det} A(m, n)$ (case: $m$ is evens
(1) Conjugation doesnt affect the det
let $\left.\overline{A(m, n)}=\left(S_{n} \otimes S_{m}\right)^{-1} A S_{n}, n\right)\left(S_{n} \otimes S_{m}\right)$
cioculant: $\left(S_{n}^{-1} \otimes S_{m}^{-1}\right)\left(I_{n} \otimes \times H_{m}+y H_{n} \otimes Z_{m}\right)\left(S_{n} \otimes S_{m}\right)$
Fact: $I_{n} \otimes \times S_{m}^{-1} H S_{m}+y S_{n}^{-1} H_{n} S_{n} \otimes S_{m}^{-1} Z_{m} S_{m}$
TM: $I_{n} \otimes \times\left(\begin{array}{llll}\lambda_{m}(1) & & \\ & \ddots & \\ & & \ddots & \\ & & \\ & & & \\ & & & (m)\end{array}\right)+Y\left(\begin{array}{lll}\lambda_{n}(1! & & \\ & & \ddots \\ & & \lambda_{n}(n)\end{array}\right) \otimes X_{m}$
Lem : $I_{n} \otimes\left(\begin{array}{lll}x \lambda_{m}(1) & \\ & & \\ & x \lambda_{m}(n)\end{array}\right)+\left(\begin{array}{lll}y \lambda_{n}(1) & & \\ & \ddots & \\ & & y \lambda_{n}(n)\end{array}\right) \otimes X_{m}$


$\overline{A(m, n)}$ il block diag w/ man blocks

each black hes exact same shape $\rightarrow$ repeat till get matrix $w / 2 \times 2$ blocks

$$
\begin{aligned}
& \text { } \operatorname{det} \overline{A(m, n)}=\prod_{b=1}^{n} \prod_{a=1}^{m / 2} \operatorname{det}\left(\begin{array}{ll}
x \lambda_{m}(a) & y \lambda_{1}(b) \\
y \lambda_{n}(b) & \left.x \lambda_{m}(n+1-a)\right)
\end{array}\right. \\
& =\prod_{b=1}^{n} \prod_{a^{=1}}^{m / 2}\left(x^{2} \lambda_{m}(a) \lambda_{m}(m+1-a)-y^{2} \lambda_{n}(b)^{2}\right) \text { apply THM } \\
& \cos (\pi-\theta)=-\cos \\
& =\prod_{b=1}^{n} \prod_{a=1}^{m / 2}\left(-4 x^{2} \cos \frac{\pi a}{m+1} \cos ^{\frac{\pi(m+1-a)}{m+1}+4 y^{2} \cos ^{2}\left(\frac{\pi b}{n+1}\right)=4^{2} \prod_{b=1}^{n} \prod_{a=1}^{m / 2}\left(x^{2} \cos ^{2}\left(\frac{n a}{m+1}\right)+y^{2} \cos ^{2}\left(\frac{\pi b}{n+1}\right)\right.}\right.
\end{aligned}
$$

by $\operatorname{Pf}(A(m, n))=\sqrt{\operatorname{det} A(m, n)}$ ploy in $x=y=1$
THM2 The \# of dimers on man lattice $=\operatorname{Pf}\left(A(m, A)=\sqrt{\operatorname{dec}(A(m, n))}=2^{m n} \prod_{b=1}^{n} \prod_{a=1}^{m / 2}\left(\cos ^{2}\left(\frac{1+a}{m+1}\right)+\cos ^{2}\left(\frac{\pi b}{n+y}\right)\right)^{1 / 2}\right.$
Ex: $m=2, n=3 \quad \#$ dimers $=8 \prod_{b=1}^{3}\left(\cos ^{2} \frac{\pi}{3}+\cos ^{2} \frac{\pi b}{4}\right)^{1 / 2}=8(3 / 8)=3$

Cinculant Upshot $\left(U_{p}\right)^{n}=1$

$$
\begin{aligned}
& {[C]\left[\begin{array}{l}
1 \\
w_{n} \\
w_{j} \\
w_{j} j_{1}
\end{array}\right]=\left[\begin{array}{l}
c_{0}+C_{1} w_{n}+\ldots+C_{n-1} w_{j}^{n-1} \\
C_{n+1}+C_{0} w_{n}+\ldots+c_{n-2} w_{j}^{n-2}
\end{array}\right]=\left[\begin{array}{l}
\lambda_{0} \\
\lambda_{1} \\
\lambda_{2} \\
\lambda_{n 1}
\end{array}\right]} \\
& E x=\left(\begin{array}{ll}
2 & 1 \\
1 & 2
\end{array}\right) \begin{array}{l}
(2-\lambda)^{2}-1=4-4 \lambda+\lambda^{2}-1 \Rightarrow \\
\lambda^{2}-4 \lambda+3 \Rightarrow(\lambda-3)(\lambda-1)
\end{array}
\end{aligned}
$$

Fram daore $\lambda_{0}=2 w_{2}^{0.0}+w_{2}^{1.0}=2 e^{0}+e^{0}=2+1=3$

$$
\lambda_{1}=2 w_{2}^{0.1}+w_{2}^{1.1}=2 e^{0}+1 e^{\frac{2 \pi 1}{2}}=2-1=1
$$

