XXX Spin Chain and Bethe Ansatz

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Disclaimer

As a physics and math double major I also typed up some physics facts because they were interesting to me, but they will most likely be skipped during the lecture.

1 Introduction

Heisenberg spin chain - proposed in 1928 by Heisenberg as toy model for studying magnetism. We solve the model by Bethe ansatz - construct the eigenstate of this model in analytic manner - invented by Bethe in 1931 - this is arguably the starting point of quantum integrability.

2 Spins

View spin as a linear space which gives a representation of the $SU(2)$ algebra, defined by the commutation relations

$$[S^\alpha, S^\beta] = i\epsilon^{\alpha\beta\gamma}S^\gamma, \alpha, \beta, \gamma = 1, 2, 3 \tag{1}$$

The simplist representation of this algebra is the Pauli matrices:

$$S^1 = \frac{1}{2}\sigma^x, S^2 = \frac{1}{2}\sigma^y, S^3 = \frac{1}{2}\sigma^z \tag{2}$$

Where

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3}$$

Denote the two basis vectors, 'spin up', and 'spin down', of this 2-dimensional vector space by:

$$|\uparrow\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |\downarrow\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4}$$

Any vector can be expressed as a linear combination of these two basis vectors.

$$v = c_1|\uparrow\rangle + c_2|\downarrow\rangle, c_1, c_2 \in \mathbb{C} \tag{5}$$

Next, we introduce two operators:

$$S^{\pm} = S^x \pm iS^y \tag{6}$$

In the Pauli matrix representation, they are given by:

$$S^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, S^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{7}$$

Acting on the basis vectors we have:

$$\begin{aligned} S^+|\uparrow\rangle = 0, S^-|\uparrow\rangle = |\downarrow\rangle, S^z|\uparrow\rangle = \frac{1}{2}|\uparrow\rangle, \\ S^+|\downarrow\rangle = |\uparrow\rangle, S^-|\downarrow\rangle = 0, S^z|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle \end{aligned} \tag{8}$$

3 Heisenberg Spin Chain

Spin chain = chain of spins. On a 1D lattice of L sites, we put a spin on each site, where they interact according to a quantum Hamiltonian.

Hilbert space Spin chain is the direct product of all spins

$$\mathcal{V} = V_1 \otimes V_2 \otimes \cdots \otimes V_L \tag{9}$$

V_k is linear space at site k. Call $\mathcal{V}$ the Hilbert space of the spin chain, a linear space of dimension 2^L. A convinent basis is given by states of the following type:

$$|\uparrow\rangle_1 \otimes |\uparrow\rangle_2 \otimes \cdots \otimes |\uparrow\rangle_L, |\downarrow\rangle_1 \otimes |\downarrow\rangle_2 \otimes \cdots \otimes |\downarrow\rangle_L, \tag{10}$$

Most of the times we omit the $\otimes$ symbol and simply write:

$$|\uparrow\rangle_1 \otimes |\uparrow\rangle_2 \otimes \cdots \otimes |\uparrow\rangle_L \equiv |\uparrow\uparrow\cdots\uparrow\rangle \tag{11}$$

Hamiltonian Hamiltonian of a Heisenberg spin chain is given by:

$$\hat{H} = \sum_{n=1}^{L}(J_x S_n^x S_{n+1}^x + J_y S_n^y S_{n+1}^y + J_z S_n^z S_{n+1}^z) \tag{12}$$

Special cases J_x, J_y, J_z specify the strength of spin interaction in each direction, here are some special cases we can consider:

1. $J_x = J_y = 0, J_z \neq 0$ (the Ising spin chain)
2. $J_z = 0, J_x = J_y \neq 0$ (the XX spin chain, equivalent to free lattice fermion by Jordan-Wigner transformation)
3. $J_x = J_y = J_z \neq 0$ (isotropic case, the XXX spin chain)
4. $J_x = J_y \neq J_z \neq 0$ (anisotropic case, XXZ spin chain)

5. $J_x \neq J_y \neq J_z \neq 0$ (completely anisotropic case, XYZ spin chain)

Interaction range In our Hamiltonian, we only consider the interaction between pairs S_n and S_{n+1}, but more generally we can consider the Hamiltonian involving k sites, in this case we say the interaction range is k. Notice that each spin operator only act on site-n and leaves the other sites unchanged, therefore they're called *local spin operators*.

Boundary condition We need to specify what S_{L+1} is since we only have L sites, here we define $S^\alpha_{L+1} \equiv S^\alpha_1$, which is a periodic boundary condition. Other boundary conditions are also possible (for example, the *twisted boundary conditions*: $S^\pm_{L+1} = \kappa^\pm S^\pm_1$ and $S^z_{L+1} = S^z_1$)

3.1 Brief history of Heisenberg spin chain

(this will **definitely** be skipped during actual presentation)

- In 1931 Hans Bethe proposed an anlytic method to construct eigenstates for XXX spin chain ($J_x = J_y = J_z$): the coordinate Behte ansatz. It can also be used to solve a large number of 1D models. (for example the antiferromagnetic ground satte in the thermodynamic limit, or the excitations on the antiferromagnetic vacuum).
- Bethe ansatz was generalized by C.N.Yang and C.P.Yang in 1966 to solve the XXZ spin chain, however it was clear new techiques were required to solve the XYZ spin chain.
- XYZ spin chain was solve in 1972 by Rodney Baxter (he studied the 2D classical lattice model, the eight-vertex model, which is equivalent to the 1D XYZ model)
- QISM / ABA, in the mid 70s, Leningrad School led by Faddeev, developed the quantum inverse scattering method (QISM) or the algebraic Bethe ansatz (ABA), this method relies heaviliy on underlying algebraic structure of integrable models.
- Quantum group: In 1985, Drinfeld come up with an algebraic structure he called the quantum group, which is a special Hopf algebra.

4 The XXX Spin Chain

Here we focus on the XXX spin chain with periodic boundary conditions. Let's write our Hamiltonian in the following form:

$$\begin{aligned} H_{XXX} &= -J\sum_{n=1}^{L}(S^x_n S^x_{n+1} + S^y_n S^y_{n+1} + S^z_n S^z_{n+1}) \\ &= -\frac{J}{2}\sum_{n=1}^{L}(S^-_n S^+_{n+1} + S^+_n S^-_{n+1} + 2S^z_n S^z_{n+1}) \end{aligned} \tag{13}$$

where

$$S_n^a = \frac{1}{2}\sigma_n^a, a = x, y, z$$
$$S_n^{\pm} = S_n^x \pm iS^y \tag{14}$$

WLOG take $J = 1$. Next we proceed to find the eigenvalues and eigenstates of this Hamiltonian. In this section we do it by brute force.

Structure of Hilbert Space Let's first decompose Hilbert space into smaller subspaces. Consider spin-ups as "vacuum" and spin-downs as "excitations". For a length-L spin chain, we decompose the Hilbert space into sectors with $0, 1, 2\cdots$ number of spin-downs. For example, for $L = 3$, we have the following sectors.

- vacuum $|\uparrow\uparrow\uparrow\rangle$
- One spin-down $|\downarrow\uparrow\uparrow\rangle, |\uparrow\downarrow\uparrow\rangle, |\uparrow\uparrow\downarrow\rangle$
- Two spin-downs $|\downarrow\downarrow\uparrow\rangle, |\uparrow\downarrow\downarrow\rangle, |\uparrow\downarrow\downarrow\rangle$
- Three spin-downs $|\downarrow\downarrow\downarrow\rangle$

This decomposition makes sense because the Hamiltonian H_{XXX} preserves the number of down spins (**check**).

5 Coordinate Bethe Ansatz

5.1 Vacuum State

The state with all spin-ups is called the *pseudo-vacuum* state, it is denoted by:

$$|\Omega\rangle = |\uparrow\uparrow\cdots\uparrow\rangle = |\uparrow^L\rangle \tag{15}$$

$|\Omega\rangle$ automatically diagonalize the Hamiltonian (in other words, $|\Omega\rangle$ is an eigenvector of our Hamiltonian), since any time we have something like $S_{n+1}^+|\uparrow\rangle$, the state vanishes, thus all the first two terms in the sum always vanish:

$$H_{XXX}|\Omega\rangle = -\sum_{n=1}^{L} S_n^z S_{n+1}^z|\Omega\rangle = -\frac{L}{4}|\Omega\rangle = E_0|\Omega\rangle \tag{16}$$

We have taken $J = 1$, and the vacuum energy is given by the constant:

$$E_0 = -\frac{L}{4} \tag{17}$$

We can 'build' the other basis vectors containing down spins by using the S_n^- operator for the nth site.

$$|n_1, n_2, \cdots, n_N\rangle = S_{n_1}^- S_{n_2}^- \cdots S_{n_N}^-|\Omega\rangle = |\cdots\downarrow^{n_1}\cdots\downarrow^{n_2}\cdots\downarrow^{n_N}\cdots\rangle \tag{18}$$

Each spin-down is called a 'magnon', a magnetic excitation. In the N-magnon sector, an eigenvector will be the linear combination of these basis vectors. Bethe's main achievement is to produce formulas for the coefficients.

5.2 One magnon sector

For one magnon sector, we simply take a linear combination of the states $|n\rangle$, where n represent the location of the spin-down vector. Notice that the Hamiltonian is invariant by translation of one site (**true for arbitrary sites too?**). This space translation invariance inspires us to propose eigenvectors that looks like the lattice version of a plane wave:

$$|\Psi(p)\rangle = \sum_{n=1}^{L} e^{ipn}|n\rangle \tag{19}$$

Physically p is the momentum of the magnon. To compute how the Hamiltonian acts on our state, we have the following relations:

$$\begin{aligned}\sum_{l=1}^{L} S_l^+ S_{l+1}^- |n\rangle &= |n+1\rangle \\ \sum_{l=1}^{L} S_l^- S_{l+1}^+ |n\rangle &= |n-1\rangle \\ \sum_{l=1}^{L} S_l^z S_{l+1}^z |n\rangle &= \frac{(L-2)-2}{4}|n\rangle = \frac{L-4}{4}|n\rangle\end{aligned} \tag{20}$$

(work out an example, using the state $|4\rangle$: $S_4^+ S_5^- |\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\rangle = |\uparrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\rangle$, all other terms in summation vanish)

Alternatively, introduce:

Lemma 5.1. $S_l^- |n_1, n_2, \cdots, n_N\rangle = \begin{cases} 0 & \textit{if } l \in n_1, n_2, \cdots, n_N \\ |n_1, n_2, \cdots, n_N, l\rangle & \textit{otherwise} \end{cases}$

$S_l^+ |n_1, n_2, \cdots, n_N\rangle = \begin{cases} 0 & \textit{if } l \notin n_1, n_2, \cdots, n_N \\ |n_1, n_2, \cdots, n_N, \hat{l}\rangle & \textit{otherwise} \end{cases}$

Now we can write:

$$\begin{aligned} H_{XXX}|\Psi(p)\rangle &= \sum_{n=1}^{L} e^{ipn} H_{XXX}|n\rangle \\ &= -\frac{1}{2}\sum_{n=1}^{L} e^{ipn}(|n+1\rangle + |n-1\rangle + \frac{L-4}{2}|n\rangle) \\ &= \frac{1}{2}(e^i p + e^{-ip} + \frac{L-4}{2})\sum_{n=1}^{L} e^{ipn}|n\rangle \\ &= -(cos(p) + \frac{L-4}{4})|\Psi(p)\rangle = E_1(p)|\Psi(p)\rangle \end{aligned} \tag{21}$$

So $|\Psi(p)\rangle$ is an eigenstate of H_{xxx} with eigenvalue $E_1(p)$. Consider:

$$\epsilon(p) = E_1(p) - E_0 = 1 - cos(p) \geq 0 \tag{22}$$

Here $\epsilon(p)$ can be understood as the energy of an excitation.

quantization condition One have one puzzle left, the momentum p was introduced by hand, can it really take any value we want? If each state corresponds to an eigenstate $|\Psi(p)\rangle$, we will have infinite number of eigenstates. On the other hand, since the Hilbert space of one magnon sector has dimension L (with L one spin-down basis states), which means we should have exactly L eigenstates. To solve this, we recall that in quantum mechanics, momentum p has to be quantized. Here the analogous quantization condition comes from the periodc boundary condition:

$$\begin{aligned}
&|n+L\rangle \equiv |n\rangle \\
&|\Psi(p)\rangle = \sum_{n=1}^{L} e^{ipn}|n\rangle = \sum_{n=1}^{L} e^{ipn}|n+L\rangle = \sum_{n=1}^{L} e^{ip(n-L}|n\rangle \\
&e^{-ipL} = \cos(pL) - i\sin(pL) = 1 \\
&pL = 2m\pi, m = 1, 2, \cdots, L
\end{aligned} \tag{23}$$

This quantization condition leads to the possible values of p:

$$p = \frac{2\pi m}{L}, m = 1, 2, \cdots, L \tag{24}$$

5.3 Two magnon sector

For the two magnon case, we take linear combination of states $|n_1, n_2\rangle$, the eigenstate takes the following form:

$$|\Psi(p_1, p_2)\rangle = \sum_{1 \leq n_1 < n_2 \leq L} \chi(\mathbf{p}|\mathbf{n})|n_1, n_2\rangle \tag{25}$$

Here we have:

$$\begin{aligned}
\mathbf{p} &= (p_1, p_2) \\
\mathbf{n} &= (n_1, n_2)
\end{aligned} \tag{26}$$

p_1,p_2 are the momenta of the two magnons.

If we have a very long spin chain, n_1 and n_2 are most often very separated from each other, so we rarely have both spins flipped when considering only the nearest neighbors in our Hamiltonian. Thus we can consider them as approximately two separate one particle states, where the wave function must contain factors like $e^{ip_1n_1+ip_2n_2}$, assuming n_1 is on the left of n_2. However the order of momenta can also be swapped, thus we have the following ansatz:

$$\chi(\mathbf{p}|\mathbf{n}) = A(p_1, p_2)e^{ip_1n_1+ip_2n_2} + A(p_2, p_1)e^{ip_2n_1+ip_1n_2} \tag{27}$$

The exact expression for $A(p_1, p_2)$ is not relevant since it's defined up to a constant (we can always normalize in the end), more important is the ratio of $A(p_1, p_2)$ and $A(p_2, p_1)$.

which can be interpreted as the scattering matrix of the two magnons with momenta p_1, p_2:

$$S(p_1, p_2) = \frac{A(p_2, p_2)}{A(p_1, p_2)} \tag{28}$$

If $|\Psi(p_1, p_2)\rangle$ is an eigenstate we must have:

$$H_{XXX} = |\Psi(p_1, p_2)\rangle = E_2(p_1, p_2)|\Psi(p_1, p_2)\rangle \tag{29}$$

In this section we want to determine both the eigenvalue $E_2(p_1, p_2)$ and the scattering matrix $S(p_1, p_2)$.

To proceed, we divide the summation in the wavefunction into two components:(1) where the spin-down states are not nearest neighbors or $n_2 \neq n_1 + 1$, (2) where they are, $n_2 = n_1 + 1$, use the following relations:

$$\begin{aligned}
&\sum_{l=1}^{L} S_l^+ S_{l+1}^- |n_1, n_2\rangle = |n_1 + 1, n_2\rangle + |n_1, n_2 + 1\rangle \\
&\sum_{l=1}^{L} S_l^- S_{l+1}^+ |n_1, n_2\rangle = |n_1 - 1, n_2\rangle + |n_1, n_2 - 1\rangle \\
&\sum_{l=1}^{L} S_l^z S_{l+1}^z |n_1, n_2\rangle = \frac{L-8}{4} |n_1, n_2\rangle
\end{aligned} \tag{30}$$

We want to have:

$$H_{XXX} \sum_{1 \leq n_1 < n_2 \leq L} \chi(\mathbf{p}|n_1, n_2)|n_1, n_2\rangle = E_2(p_1, p_2) \sum_{1 \leq n_1 < n_2 \leq L} \chi(\mathbf{p}|n_1, n_2)|n_1, n_2\rangle \tag{31}$$

This means the coefficients of each $|n_1, n_2\rangle$ need to agree on both sides.

Distant magnons First consider the case where n_1, n_2 are not nearest neighbors. Comparing the two sides, we find:

$$\begin{aligned}
E_2\chi(\mathbf{p}|n_1, n_2) = &-\frac{1}{2}[\chi(\mathbf{p}|n_1 - 1, n_2) + \chi(\mathbf{p}|n_1, n_2 - 1)] \\
&- \frac{1}{2}[\chi(\mathbf{p}|n_1 + 1, n_2) + \chi(\mathbf{p}|n_1, n_2 + 1)] - \frac{L-8}{4}\chi(\mathbf{p}|n_1, n_2)
\end{aligned} \tag{32}$$

The intuition for this is that our operator combinations can only make the down-spin move one lattice to the left or right, or leave them unchanged. Thus, the only initial states that generate the $|n_1, n_2\rangle$ state when acted on by our operator combinations are:

$$|n_1 \pm 1, n_2\rangle, |n_1, n_2 \pm_1\rangle, |n_1, n_2\rangle \tag{33}$$

Next, using the explicit form of $\chi(\mathbf{p}|n_1, n_2)$, we have:

$$\begin{aligned}
\chi(\mathbf{p}|n_1 + 1, n_2) + \chi(\mathbf{p}|n_1, n_2 + 1) = (e^{ip_1} + e^{ip_2})\chi(\mathbf{p}|n_1, n_2) \\
\chi(\mathbf{p}|n_1 - 1, n_2) + \chi(\mathbf{p}|n_1, n_2 - 1) = (e^{-ip_1} + e^{-ip_2})\chi(\mathbf{p}|n_1, n_2)
\end{aligned} \tag{34}$$

Thus we have:

$$\begin{aligned} E_2(p_1,p_2) &= -\frac{1}{2}((e^{ip_1}+e^{ip_2})+(e^{-ip_1}+e^{-ip_2})) - \frac{L-8}{4} \\ E_2 - E_0 &= 2-\cos(p_1)-\cos(p_2) = \epsilon(p_1)+\epsilon(p_2) \end{aligned} \tag{35}$$

We can see the energy is the sum of energy of the two magnons. Although we derived this relation by assuming the sites are not next to one another, it also applies in the general case. **check?**

Neighboring magnons Consider the case when two magnons are next to each other, specifically, consider $\chi(\mathbf{p}|n_1, n_1+1)$.

We have:

$$\begin{aligned} \sum_k S_k^+ S_{k+1}^- |n_1, n_1+1\rangle &= |n_1, n_1+2\rangle \\ \sum_k S_k^- S_{k+1}^+ |n_1, n_1+1\rangle &= |n_1-1, n_1+1\rangle \\ \sum_k S_k^+ S_{k+1}^- |n_1, n_1+1\rangle &= \frac{L-4}{4}|n_1, n_1+1\rangle \end{aligned} \tag{36}$$

Similar to the previous argument, we have:

$$\begin{aligned} E_2\chi(\mathbf{p}|n_1,n_1+1) = &-\frac{1}{2}(\chi(\mathbf{p}|n_1-1,n_1+1)+\chi(\mathbf{p}|n_1,n_1+2)) \\ &-\frac{L-4}{4}\chi(\mathbf{p}|n_1,n_1+1) \end{aligned} \tag{37}$$

Relation 32 also tells us:

$$\begin{aligned} E_2\chi(\mathbf{p}|n_1,n_1+1) = &-\frac{1}{2}[\chi(\mathbf{p}|n_1-1,n_1+1)+\chi(\mathbf{p}|n_1,n_1)] \\ &-\frac{1}{2}[\chi(\mathbf{p}|n_1+1,n_1+1)+\chi(\mathbf{p}|n_1,n_1+2)] \\ &-\frac{L-8}{4}\chi(\mathbf{p}|n_1,n_1+1) \end{aligned} \tag{38}$$

Take the difference of two equations:

$$\frac{1}{2}\left(\chi(\mathbf{p}|n_1,n_1)+\chi(\mathbf{p}|n_1+1,n_1+1)\right)-\chi(\mathbf{p}|n_1,n_1+1)=0 \tag{39}$$

Solve this equation for $S(p_1,p_2)$ **actually try this**:

$$S(p_1,p_2) = -\frac{1-2e^{ip_2}+e^{i(p_1+p_2)}}{1-2e^{ip_1}+e^{i(p_1+p_2)}} \tag{40}$$

By trig identities, we have:

$$S(p_1,p_2) = -\frac{\frac{1}{2}\cot\frac{p_1}{2}-\frac{1}{2}\cot\frac{p_2}{2}-i}{\frac{1}{2}\cot\frac{p_1}{2}-\frac{1}{2}\cot\frac{p_2}{2}+i} \tag{41}$$

Next, we use again the periodic boundary condition to quantize the momenta:

$$|n_1, n_2\rangle = |n_2 - L, n_1\rangle \tag{42}$$

So for the wave function we have:

$$\begin{aligned}
|\Psi(p_1, p_2)\rangle &= \sum_{n_1<n_2} \chi(\mathbf{p}|n_1, n_2)|n_1, n_2\rangle = \sum_{n_1<n_2} \chi(\mathbf{p}|n_1, n_2)|n_2 - L, n_1\rangle \\
&= \sum_{n_1'<n_2'} \chi(\mathbf{p}|n_2', n_1' + L)|n_1', n_2'\rangle \\
&= \sum_{n_1<n_2} \chi(\mathbf{p}|n_2, n_1 + L)|n_1, n_2\rangle
\end{aligned} \tag{43}$$

In the second line we defined $n_1' = n_2 - L$ and $n_2' = n_1$ and in the third line we renamed the dummy indices. Comparing the first and third line we find:

$$\chi(\mathbf{p}|n_1, n_2) = \chi(\mathbf{p}|n_2, n_1 + L) \tag{44}$$

Plugging in the explicit form of $\chi(\mathbf{p}|n_1, n_2)$ and identifying coefficients of exponential social factors we have:

$$e^{ip_1L}S(p_1, p_2) = 1, e^{ip_2L} = S(p_1, p_2) \tag{45}$$

Using the explicit form of the S-matrix, we find:

$$S(p_1, p_2)S(p_2, p_1) = 1 \tag{46}$$

Which allow us to rewrite the relations in a more symmetric form:

$$e^{ip_1L}S(p_1, p_2) = 1, e^{ip_2L}S(p_2, p_1) = 1 \tag{47}$$

This set of quantization conditions is called the Bethe conditions.

5.4 Three magnon sector

The one and two magnon cases somewhat follows from quantum mechanics. The crucial assumption is made in the 3 magnons case:

$$|\Psi(p_1, p_2, p_3)\rangle = \sum_{1\leq n_1<n_2<n_3\leq L} \chi(\mathbf{p}|\mathbf{n})|n_1, n_2, n_3\rangle \tag{48}$$

Now, again, we want to propose a form for the wave function $\chi(\mathbf{p}|n)$. We follow the same general idea of the one and two magnons cases, however, here we have $3! = 6$ possibilites of permuting the momenta $\mathbf{p} = p_1, p_2, p_3$. Let's denote the permutations of $1, 2, 3$ by σ. The set of all possible permutations is denoted by S_3. We have $\sigma \in S_3$:

$$S_3 = \{123, 213, 132, 213, 231, 312, 321\} \tag{49}$$

For each σ, we can define the corresponding permutation of momenta p_σ and the coefficient A:

$$\sigma = \{321\}, \mathbf{p}_\sigma = \{p_3 p_2 p_1\}, A(\mathbf{p}_\sigma) = A(p_3, p_2, p_1) \tag{50}$$

Then we can write down the proposal for the wave function:

$$\chi(\mathbf{p}|\mathbf{n}) = \sum_{\sigma \in S_3} A(\mathbf{p}_\sigma e^{i(p_{\sigma(1)}n_1 + p_{\sigma(2)}n_2 + p_{\sigma(3)}n_3)}) \tag{51}$$

Normally we would follow the same strategy as in the two magnon sector, applying the eigenvalue equation and find all undetermined quantities. However, here we have 6 unknown ratios to determine, which is a lot. Instead, we make a conjecture based on physical intuitions.

We start with the eigenvalue, in the one magnon sector, we had:

$$E_1(p) - E_0 = \epsilon(p), \epsilon(p) = 1 - \cos(p) \tag{52}$$

and in the two magnon sector:

$$E_2(p) - E_0 = \epsilon(p_1) + \epsilon(p_2) \tag{53}$$

So now the natrual conjecture for the three magnon case is:

$$E_3(p) - E_0 = \epsilon(p_1) + \epsilon(p_2) + \epsilon(p_3) \tag{54}$$

Which indeed turns out to be true.

It's slightly more non-trivial to guess the form of $A(p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)})/A(p_1, p_2, p_3)$. In the two magnon case, this has the physical interpretation of the S-matrix: if we exchange the momentum we have to pay an S-matrix $S(p_1, p_2)$. The crucial insight of Bethe: the ratio $A(\mathbf{p}_\sigma)/A(\mathbf{p})$ *factorizes into a sequence of two-body S-matrices.* Any $A(\mathbf{p}_\sigma)$ can be brought into $A(\mathbf{p})$ by exchanging two particles at a time. We propose that each time we exchange two particles, we pay with an S-matrix by the following rule:

$$A(\cdots, p_j, p_k, \cdots) = S(p_k, p_j) A(\cdots, p_k, p_j, \cdots), \text{if } j > k \tag{55}$$

Using this rule recursively, we have, for example:

$$\begin{aligned} A(p_3, p_2, p_1) &= S(p_1, p_2) A(p_3, p_1, p_2) = S(p_1, p_2) S(p_1, p_3) A(p_1, p_3, p_2) \\ &= S(p_1, p_2) S(p_1, p_3) S(p_2, p_3) A(p_1, p_2, p_3) \end{aligned} \tag{56}$$

Where $S(p_i, p_j)$ is the two-body S-matrix which we have fixed in the two magnon sector.

Therefore we find the ratio:

$$\frac{A(p_3, p_2, p_1)}{A(p_1, p_2, p_3)} = S(p_1, p_2) S(p_1, p_3) S(p_2, p_3) \tag{57}$$

This gives us a complete proposal for the eigenvector and eigenvalue in the three magnon sector, with the corresponding quantization condition obtained from:

$$\chi(\mathbf{p}n_1, n_2, n_3) = \chi(\mathbf{p}|n_2, n_3, n_1 + L) \tag{58}$$

Which leads to the following three equations:

$$\begin{aligned} e^{ip_1L}S(p_1, p_2)S(p_1, p_3) = 1 \\ e^{ip_2L}S(p_2, p_1)S(p_2, p_3) = 1 \\ e^{ip_3L}S(p_3, p_1)S(p_3, p_2) = 1 \end{aligned} \tag{59}$$

5.5 N-magnon sector

Now we're ready to write down the formula for an arbitrary N:

$$|\Psi(\mathbf{p})\rangle = \sum_{1\le n_1<\cdots<n_N\le L}\sum_{\sigma\in S_N}\frac{A(\mathbf{p}_\sigma)}{A(\mathbf{p})}e^{i(p_{\sigma(1)}n_1+p_{\sigma(2)}n_2+\cdots+p_{\sigma(N)}n_N)}|n_1,\cdots,n_N\rangle \tag{60}$$

The eigenvalue is given by the equation:

$$H_{XXX}|\Psi(\mathbf{p})\rangle = E_N(\mathbf{p})|\Psi(\mathbf{p})\rangle \tag{61}$$

where

$$E_n(\mathbf{p}) = E_0 + \sum_{k=1}^{N}\epsilon(p_k) \tag{62}$$

It's useful to change variables from momenta to the so-called rapidities:

$$e^{ip_k} = \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}, u_k = \frac{1}{2}\cot\frac{p_k}{2} \tag{63}$$

Now the BAE can be written in a simpler form:

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L \prod_{j\neq k}^{N}\frac{u_k - u_j - i}{u_k - u_j + 1} = 1, k = 1, 2, \cdots, N \tag{64}$$

And the energy of each magnon also take a simpler form:

$$\epsilon(p_k) = 1 - \cos(p_k) = \frac{2}{4u_k^2 + 1} \tag{65}$$

6 Concluding Remarks

Basically, we have transformed the problem in linear algebra (looking for eigenstates and eigenvalues of a matrix) into a problem of solving the algebraic (Bethe) equations. Why is this useful?

At a technical level, Bethe equations can be used for the approximation of large L and M cases, where brute force diagonalization is impractical. Additionally, this provides as a physical picture of magnons propagating on top of the vacuum and interacting with each other in an integrable way.