Notes for Math Seminar

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1 Models with $\mathbf{Z}_n$ Symmetry

We have that the in the $\mathbb{Z}_n$ model, spins are planar vectors of unitary length can be identifed by discrete angels θ_i with respect to the horizontal axes:

$$a^{(k)} = \frac{2\pi k}{n}, k = 0, 1, 2, ...n - 1$$

We then have that the hamiltonian for this system is:

$$\begin{aligned} \mathcal{H} &= -\mathcal{J} \sum_{<ij>} \overrightarrow{S}_i \cdot \overrightarrow{S}_j \\ &= -\mathcal{J} \sum_{<ij>} \cos(\theta_i - \theta_j) \end{aligned}$$

We have that this system is invariant under the transformation $k \to k + m($ mod $n)$. For $n = 2$, we get the regular Ising Model or the 2-state Potts model. When $n = 3$, we have 3-state Potts model. When $n \to \infty$ we have that $\mathbb{Z}_n$ becomes equivalent to $O(2)$ model. We have for the 1-D Model:

$$Z_n = \sum_{\theta_1=0}^{n-1} ... \sum_{\theta_N=0}^{n-1} \exp\left(J \sum_{i=0}^{N-1} \cos\left(\frac{2\pi}{n}(\theta_i - \theta_{i+1})\right)\right)$$

For $N = 1$, we have that $Z_N = n$ (number of possible states). We then find that the rest of the partition function using recursive method:

$$Z_{N+1} = Z_N \sum_{\theta_{N+1}=0}^{n-1} \exp\left(J \cos\left(\frac{2\pi}{n}(\theta_N - \theta_{N+1})\right)\right)$$

We then have that we can say $\mu_1(J, n) = \sum_{\theta_{N+1}=0}^{n-1} \exp\left(J \cos\left(\frac{2\pi}{n}(\theta_N - \theta_{N+1})\right)\right)$. We have that $Z_N = n[\mu_1(J, n)]^{N-1}$. We then have that the correlation function of two spins is:

$$\begin{aligned} G(r) &= \langle \overrightarrow{S}_i \cdot \overrightarrow{S}_{i+r} \rangle \\ &= \langle \cos(\theta_i - \theta_{i+r}) \rangle \end{aligned}$$

We have that:

$$\langle S_i \cdot S_{i+r}\rangle = Z_N^{-1} \sum_{\{S_1,\ldots S_N\}} S_k \cdot S_{k+r} \exp\left(\sum_{i=0}^{N-1} J S_i \cdot S_{i+1}\right)$$

If we then use the fact that $\sum_{\{S\}} \overrightarrow{S} e^{JS\cdot S'} = \mu_2(J,n)S'$, we have that:

$$\langle \overrightarrow{S}_i \cdot \overrightarrow{S}_{i+r}\rangle = \left(\frac{\mu_2(J,n)}{\mu_1(J,n)}\right)^r$$

2 Proof of the Cosecant Identity

We have that:

$$\begin{aligned}\Gamma(s)\Gamma(1-s) &= \pi \csc(\pi s)\\ &= \frac{\pi}{\sin(\pi s)}\end{aligned}$$

We have that our definition is:

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}$$

So we have:

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= \Gamma\left(\frac{1}{2}\right)^2\\ \Gamma(s)\Gamma(1-s) &= \int_0^\infty e^{-x}x^{s-1}dx \int_0^\infty e^{-y}y^{-s-1}dy\\ &= \int_0^\infty \int_0^\infty e^{-x}x^{s-1}e^{-y}y^{-s}dydx\end{aligned}$$

We do a change of variables $u = \frac{x}{y}$, with $du = \frac{dx}{y}$, $dx = ydu$, so that we have that:

$$\begin{aligned}\Gamma(s)\Gamma(1-s) &= \int_0^\infty \int_0^\infty e^{-uy}(uy)^{s-1}e^{-y}y^{-s}dydx\\ &= \int_0^\infty u^{s-1}\int_0^\infty e^{-(u+1)y}dydu\\ &= \int_0^\infty \left[u^{s-1}\left[\frac{e^{-(u+1)y}}{u+1}\right]\right]_0^\infty du\\ &= \int_0^\infty \frac{u^{s-1}}{u+1}du\end{aligned}$$

We then have that we can use geometric series to solve this integral:

$$\begin{aligned}\frac{1}{1+u} &= \frac{1}{1-(-u)} \\ &= \sum_{n=0}^{\infty}(-u)^n\end{aligned}$$

Where we have that $u \in [0,1)$. We can then take:

$$\int_0^{\infty} \frac{u^{s-1}}{u+1} du = \int_0^1 \frac{u^{s-1}}{u+1} du + \int_1^{\infty} \frac{u^{s-1}}{u+1} du$$

We make the variable change of the form $u = \frac{1}{v}$, and we have that $dv = -\frac{du}{u^2}$, and we have that $du = -u^2 dv$. We then have that:

$$\begin{aligned}\int_1^{\infty} \frac{u^{s-1}}{u+1} du &= -\int_1^0 \frac{\left(\frac{1}{v}\right)^{s-1}}{\frac{1}{v}+1} \frac{dv}{v^2} \\ &= \int_0^1 \frac{v^{-s}}{v+1} dv\end{aligned}$$

So we then have that:

$$\begin{aligned}\Gamma(s)\Gamma(1-s) &= \int_0^1 \frac{u^{s-1}}{u+1} du + \int_0^1 \frac{v^{-s}}{v+1} dv \\ &= \sum_{n=0}^{\infty}(-1)^n \int u^{n+s-1} du + \sum_{n=0}^{\infty}(-1)^n \int_0^1 v^{n-s} dv \\ &= \sum_{n=0}^{\infty}(-1)^n \left[\frac{u^{n+s}}{n+s}\right]_0^1 + \sum_{n=0}^{\infty} \left[(-1)^n \frac{v^{n-s+1}}{n-s+1}\right]_0^1 \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+s} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n-s+1} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k}{k+s} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k-s} + \frac{1}{s} \\ &= \frac{1}{s} + \sum_{k=1}^{\infty}(-1)^k \left[\frac{1}{k+s} - \frac{1}{k-s}\right] \\ &= \frac{1}{s} - \sum_{k=1}^{\infty}(-1)^k \frac{2s}{k^2-s^2}\end{aligned}$$

We then have that:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

If s is an integer both sides are infinite, so we most likely have that this is of the form $\frac{c}{\sin(\pi s)}$. If $s = \frac{1}{2}$, we have:

$$\begin{aligned}
\frac{1}{s} - \sum_{k=1}^{\infty}(-1)^k \frac{2s}{k^2 - s^2} &= 2 - \sum_{k=1}^{\infty}(-1)^k \frac{1}{k^2 - \left(\frac{1}{2}\right)^2} \\
&= 2 - \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 - \frac{1}{4}} \\
&= 2 - 4\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)(2k+1)} \\
&= 2 - 4\sum_{k=1}^{\infty} \frac{(-1)^k}{2} \left(\frac{1}{2k-1} - \frac{1}{2k+1}\right) \\
&= 2 + 2\left(\frac{1}{1} - \frac{1}{3}\right) - 2\left(\frac{1}{3} - \frac{1}{5}\right) + ... \\
&= 4\left(1 - \frac{1}{3} + \frac{1}{5} - ...\right) \\
&= 4\frac{\pi}{4}
\end{aligned}$$

This is using the Gregory-Leibniz formula $\left(1 - \frac{1}{3} + \frac{1}{5} - ...\right) = \frac{\pi}{4}$. As a result, we have that:

$$\frac{c}{\sin\left(\frac{\pi}{2}\right)} = c = \frac{\pi}{4} \cdot 4 = \pi$$