SYMMETRIC SPACES

CALEB JI

Abstract. These are very informal notes. Everything here is very standard.

1. Basic definitions and examples

Symmetric spaces were initially studied by Élie Cartan. This was after his work classifying Lie algebras, and he found deep links between these two subjects.

In terms of Riemannian geometry, a locally symmetric space is one with a local geodesic symmetry which is an isometry at each point, while a symmetric space is a locally symmetric space whose geodesic symmetries can be extended to the whole space. Some simple examples include Euclidean space and the n-sphere. The quotient of the upper-half plane by $\mathrm{SL}(2, \mathbb{Z})$ is a locally symmetric space.

One can also characterize symmetric spaces via Lie theory. Indeed, symmetric spaces often arise in the form $\mathbb{R}^n \times G/K$ with G a semisimple Lie group and K a maximal compact subgroup. From an algebraic point of view, one begins with a semisimple Lie group G with an involution σ, and an open subgroup H of G^σ; then the associated symmetric space is given by G/H. If we start with a symmetric space M using the geometric notion, then G is given by the connected component of the isometry group of M, and K is given by the stabilizer of a point.

Let's look at some Lie-theoretic examples. If we begin with $\mathrm{GL}(n, \mathbb{R})$ and the involution $\Theta(g) = (g^T)^{-1}$, then the fixed point set is the maximal compact subgroup $O(n)$, and the quotient $\mathrm{GL}(n, \mathbb{R})/\mathcal{O}(n) \cong \mathrm{Sym}^+(n, \mathbb{R})$ (symmetric positive-definite matrices).

We could also begin with $\mathrm{Sp}(2n, \mathbb{R})$; with the same involution we get $K = U(n)$, and the quotient is known as the Siegel upper-half space $\mathbb{H}_n$, which consists of a real symmetric matrix $+i\cdot$ a positive-deifnite real symmetric matrix. Note how this generalizes $\mathbb{H}$, which is the case of $n = 1$.

2. Classification

By work of Cartan, the simply connected symmetric spaces are decomposed into $\mathbb{R}^n \times M_+ \times M_-$, with M_+ having nonnegative sectional curvature and being of compact type, and M_- having non-positive sectional curvature and being of non-compact type.

For example, the examples of quoteints above are of non-compact type, and have compact duals given by certain Grassmannians. In general, they are given by the classification of real Lie groups, along with suitable quotients.

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