SCHUR-WEYL DUALITY

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Abstract. THese are very informal notes. Everything here is very standard.

1. Representation theory of S_n

Conjugacy classes of S_n are given by partitions of n.

The classical way to construct an irreducible representation of S_n corresponding to a partition λ is through Specht modules. The Specht module V_λ is defined as follows. Note that every irrep of S_n must necessarily be of the form $\mathbb{C}[S_n]\lambda_n$ for some choice of $\lambda_n \in \mathbb{C}[S_n]$. It remains to define c_λ. To do this, take a standard Young tableaux and let a_λ be the sum of the basis elements in $\mathbb{C}[S_n]$ that preserve each row of λ, and b_λ similarly for the columns. Then set $c_\lambda := a_\lambda b_\lambda$.

Alternatively, one can define a tabloid associated to λ to be an equivalence class of labelings of λ where the ordering of the rows doesn't matter. Tabloids naturally give an S_n-representation which we denote M^λ. Then given a tableau T, we define e_T to be the class of

$$\sum_{w \in C(T)} \operatorname{sgn}(w) w(T)$$

in M^λ, where $C(T)$ is the subgroup preserving the columns. Then beginning with any T (eg the standard one) we can define the Specht module S^λ to be the subrepresentation of M^λ generated by e_T. Then $S^\lambda = V_\lambda$ is the irreducible representation associated to λ.

It turns out that if we take standard Young tableaux, these e_T form a basis of S^λ. Thus the dimension of S^λ is given by the number of these SYTs, which is given by the hook length formula:

$$\dim S^\lambda = \prod_{(i,j) \in \lambda} \frac{n!}{h_{ij}}$$

where h_{ij} refers to the hook length of each cell in the tableaux.

2. Representation theory of $\mathrm{GL}(n, \mathbb{C})$

Recall that $\mathrm{GL}(n, \mathbb{C})$ is reductive but not semisimple, as is its Lie algebra $\mathfrak{gl}(n, \mathbb{C})$. We have the Levi decomposition

$$\mathfrak{gl}(n, \mathbb{C}) = (n, \mathbb{C}) \oplus \mathbb{C}\mathbb{I}_\ltimes;$$

this is the Levi decomposition into the semisimple Levi factor and the radical.

Recall how we classified irreps of semisimple Lie algebras using the theorem of the highest weight. The root system of $(n, \mathbb{C})$ is given by $\{e_i - e_j\}$, while the weight lattice is given by $\{(\lambda_1, \dots, \lambda_n)\}$ with $\lambda_i - \lambda_j \in \mathbb{Z}$. The set of dominant roots is obtained when these differences are all non-negative for $i < j$. Since $(1, \dots, 1)$ is equal to 0 in $\mathfrak{h}^*$, the set of dominant weights is given by partitions with at most $n-1$ parts. There is a "hook-content formula" for the dimensions of these irreps:

$$\dim S'^\lambda = \prod_{(i,j) \in \lambda} \frac{n - i + j}{h_{ij}}$$

If we are interested in $\mathfrak{gl}(n, \mathbb{C})$ instead, we get to add vectors of the form $(k, \dots, k)$ and get new irreps. The classification of irreps of the Lie groups is the same.

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3. Schur-Weyl duality for $\mathrm{GL}(n, \mathbb{C})$

There is a correspondence between irreps of S_k and $\mathrm{GL}(n, \mathbb{C})$ found in $V^{\otimes k}$, where V is the standard representation of $\mathrm{GL}(n, \mathbb{C})$.

Note that $V^{\otimes k}$ is a representation of both groups, and that these group actions commute with each other.

The abstract statement of Schur-Weyl duality is that the algebras of operators which S_k and $\mathrm{GL}(n, \mathbb{C})$ generate are centralizers of each other. Concretely, we get a decomposition

$$V^{\otimes k} = \bigoplus_{|\lambda|=k} V_\lambda \otimes S_\lambda V,$$

where $S_\lambda V = \mathrm{Hom}_{S_k}(V_\lambda, V^{\otimes k})$ is the Schur functor which is an irreducible representation of $\mathrm{GL}(n, \mathbb{C})$ or is 0.

What does this mean? Well, first we get an explicit relation between the various irreps of $\mathrm{GL}(n, \mathbb{C})$ with those of S_k as n and k vary, a correspondence that we might hope eists given the abstract fact that they are both indexed by partitions. Furthermore, it exhibits Schur functors, which generalize symmetric and alternating powers.